Derivatives

<u>Defn</u>. Let $f : A (\subset \mathbb{R}^n) \longrightarrow \mathbb{R}^m$.
Suppose that A contains a neighbourhood (nbhd) of some point $x \in A$. Then the <u>directional derivative</u> of f at x with respect to a fixed vector u is defined by

$$f'(x; u) := \lim_{t \to 0} \frac{f(x+tu) - f(x)}{t},$$

provided this limit exists.

Example

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$f(x, y) = x + y + xy$

The D.D. of f at $x = (x_1, x_2)$ with respect to $u = (1, 0)$.

$$f'(x; u) = \lim_{t \to 0} \frac{f(x + tu) - f(x)}{t}$$

$$= \lim_{t \to 0} \frac{f(x_1 + t, x_2) - f(x_1, x_2)}{t}$$

$$= \lim_{t \to 0} \frac{x_1 + t + x_2 + (x_1 + t)x_2 - (x_1 + x_2 + x_1 x_2)}{t}$$

$$= \lim_{t \to 0} \frac{t + t x_2}{t} = 1 + x_2$$

Derivative of a real-valued function

Let $f: A(\subset \mathbb{R}^n) \longrightarrow \mathbb{R}$, and let A contain a nbhd of a point $x \in A$. Then f is said to be <u>differentiable</u> at x if $\exists$ a number λ such that

$$\frac{f(x+t) - f(x) - \lambda t}{t} \longrightarrow 0, \quad \text{as } t \longrightarrow 0 \qquad (*)$$

In the event that $(*)$ holds, then the unique (why?) number λ is called the <u>derivative</u> of f at x and is denoted by $f'(x)$.

Generalized derivative

Defn. Let $f: A (\subset \mathbb{R}^n) \longrightarrow \mathbb{R}^m$, and let A contain a nbhd of a point $x \in A$. We say f is <u>differentiable</u> at x if $\exists$ an $m \times n$ matrix B such that

$$\frac{f(x+h) - f(x) - \overset{\overset{m\times n}{\downarrow}}{B} \cdot \overset{\overset{n\times 1}{\downarrow}}{h}}{|h|} \to 0 \quad \text{as } h \to 0 \qquad (*)$$

If $(*)$ holds, the unique (why?) matrix B is called the derivative of f at x and is denoted by $Df(x)$.

$$B \cdot h = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ & \ddots & \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$$

linear map

Why unique?

Suppose that C is another $m \times n$ matrix satisfying (*).

Then:

$$\frac{f(x+h) - f(x) - C \cdot h}{|h|} \to 0, \text{ as } h \to 0. \quad (**)$$

$(*) - (**) \Longrightarrow$

$$\frac{(C-B)h}{|h|} \longrightarrow 0 \quad \text{as } h \to 0$$

$\hookrightarrow (A)$

Then (A) is equivalent to:

$\lim\limits_{t \to 0} \frac{(C-B)(tu)}{|t|}$ ✓✓, where u is a unit vector and $h = tu$.

$\Longrightarrow (C-B)u = 0$ ✓

$$\left(\begin{aligned} &\lim_{t \to 0} \frac{t(C-B)(u)}{|t|} \\ &= \lim_{t \to 0^+} \frac{t}{t}(C-B)u \\ &= \lim_{t \to 0^-} \frac{t}{-t}(C-B)u \end{aligned} \right)$$

But since the choice of u is arbitrary, we have

$$C = B \quad \blacksquare$$

<u>Example</u>

Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be defined by

$$f(x) = B.x + b \ , \quad b \in \mathbb{R}^n.$$

Then for each $y \in \mathbb{R}^n$, we have

$$\lim_{h \to 0} \frac{f(y+h) - f(y) - B.h}{|h|}$$

$$= \lim_{h \to 0} \frac{B.(y+h)+b-(B.y+b)-B.h}{|h|}$$

$$= 0$$

$\Rightarrow$ By definition,

$$Df(y) = B.$$

Theorem. Let $f: A (\subset \mathbb{R}^n) \to \mathbb{R}^m$. If $Df(x)$ exists, then $f'(x;u)$ exists at each u and $f'(x;u) = Df(x).u.$

Proof. Exercise

Does the converse of the above theorem hold?

No.

Example.

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2} & , \text{ if } (x,y) \neq (0,0) \\ 0 & , \text{ if } (x,y) = (0,0) \end{cases}$$

Consider

$$f'(0;u) = \lim_{t \to 0} \frac{f(0+tu) - f(0)}{t}$$

Let $u = (u_1, u_2)$

Then

$$f'(0;u) = \lim_{t \to 0} \frac{f(tu_1, tu_2) - f(0,0)}{t}$$

$$= \lim_{t \to 0} \frac{\left(\frac{t^3 u_1^2 u_2}{t^4 u_1^4 + t^2 u_2^2}\right)}{t}$$

$$= \lim_{t \to 0} \frac{u_1^2 u_2}{t^2 u_1^4 + u_2^2}$$

$$\Rightarrow \quad f'(0;u) = \begin{cases} \frac{u_1^2}{u_2}, & \text{if } u_2 \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

$f'(0;u)$ exists for all $u \neq 0$.

However, $Df(0)$ does not exist.

For if it does, then $Df(0)$ is a 1×2 matrix $[a, b]$

$$\Rightarrow f'(0,u) = Df(0) \cdot u$$

$$= [a, b] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= a u_1 + b u_2,$$

which is a linear function $\#$

Theorem. Let $f : A\,(\subset \mathbb{R}^n) \to \mathbb{R}^m$. If $Df(x)$ exists, then f is continuous at x.

Proof

For $h \neq 0$ and near 0, we write

$$f(x+h) - f(x) = |h| \overset{(*)}{\left[\frac{f(x+h) - f(x) - Df(x)h}{|h|}\right]} + Df(x)h$$

$$\Rightarrow \lim_{h \to 0} f(x+h) - f(x) = \lim_{h \to 0} |h| \left[\ \cdot \overset{(*)}{\cdot\ \cdot} \ \cdot\ \right] + 0$$

$$= 0$$

$$\Rightarrow \lim_{h \to 0} f(x+h) = f(x)$$

$\Rightarrow$ f is continuous at x.

$x = (x_1, \ldots x_n)$

$y = (y_1, \ldots y_n)$

$$\|x-y\|_2 = \sqrt{(x_1-y_1)^2 + \ldots + (x_n-y_n)^2}$$

$$\|x\|_2 = \sqrt{x_1^2 + \ldots + x_n^2}$$

$\| \ \|_\infty$ is the sup norm

$$\underline{\|x\|_\infty} = \max\{|x_1|, \ldots, |x_n|\}$$

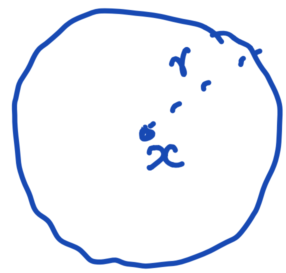
$\Rightarrow$ preferred norm

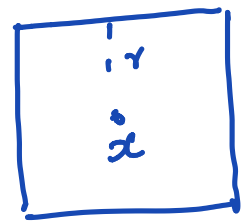
$$\|x\|_K = \sqrt[K]{(x_1)^K + \ldots (x_n)^K}$$

$$d_2(x,y) = \|x-y\|_2$$

$$d_\infty(x,y) = \|x-y\|_\infty$$

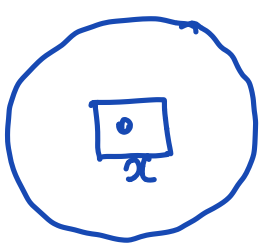
$x \in \mathbb{R}^2$

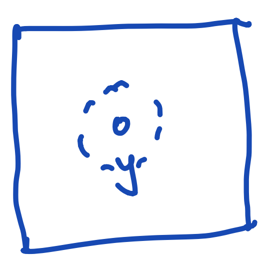
$B_{d_2}(x,r) =$ 

$B_{d_\infty}(x,r) =$ 

d_2 induces the standard topology in $\mathbb{R}^n$, while d_∞ also induces the same topology.

This is because the metric spaces are equivalent.





Unless mentioned otherwise,

$|h|$ – sup norm on h

<u>Defn</u>. Let $f : A (\subset \mathbb{R}^n) \longrightarrow \mathbb{R}$

Then the <u>j^{th} partial derivative</u> of at x (denoted by $D_j f(x)$) is defined by:

$$D_j f(x) := f'(x; e_j)$$

<u>Theorem</u>. Let $f : A (\subset \mathbb{R}^n) \longrightarrow \mathbb{R}$.

If $Df(x)$ exists, then

column vector

$$Df(x) = [D_1 f(x), \ldots, D_n f(x)].$$

<u>Proof</u>. Exercise. (Follows directly from defn)

Theorem. Let $f : A (\subset \mathbb{R}^n) \longrightarrow \mathbb{R}^m$, and let A contain a nbhd of the point x. Let $f_i : A \longrightarrow \mathbb{R}$ be the i^{th} component function of f so that

$$f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix}$$

(a) f is differentiable at x $\Longleftrightarrow$ each f_i is differentiable at x.

(b) If $Df(x)$ exists, then

$$Df(x) = \begin{bmatrix} Df_1(x) \\ \vdots \\ Df_m(x) \end{bmatrix}, \text{ where}$$

the $(i,j)^{th}$ entry of $Df(x)$ is given by $D_j f_i(x)$.

Proof. Exercise (Follows directly from defn)

Defn. Let $f : A (\subset \mathbb{R}^n) \to \mathbb{R}^m$.
If the partial derivatives of the component functions f_i of f exist at x, then the matrix

$$Jf(x) = \begin{bmatrix} D_1 f_1(x) & \cdots & D_n f_1(x) \\ \vdots & & \vdots \\ D_1 f_m(x) & \cdots & D_n f_m(x) \end{bmatrix}$$

is called the Jacobian matrix of f at x.

Note: If $Df(x)$ exists, then

$$Df(x) = Jf(x)$$

Continuously differentiable functions

$f : A (\subset \mathbb{R}^n) \to \mathbb{R}^m$

<u>Theorem</u>. Let $A \subset \mathbb{R}^n$ be open. Suppose that $D_j f_i(x)$ exists at each $x \in A$ and the $D_j f_i$ are continuous on A. Then $Df(x)$ exists at each $x \in A$.

<u>Defn</u>. A function f as in the hypothesis of the above theorem is said to be <u>continuously differentiable</u> on A. (i.e. of class C^1 on A).

<u>Defn</u>. Let $f : A (\subset \mathbb{R}^n) \longrightarrow \mathbb{R}^m$.
If the partial derivatives of f_i of all orders $\leq r$ exist and are continuous on A, then we say <u>f is of class C^r on A</u>.

$$\left(\underset{\text{2nd order}}{D_j D_i f(x)}, \ \underset{\text{3rd order}}{D_k D_j D_i f(x)} \ \cdots \right)$$

<u>Proof</u>
It suffices to show that each component of f is diff (i.e. $Df_i(x)$ exists for each i).

This means that we can restrict our attention to $f : A \longrightarrow \mathbb{R}$.

We are given that $D_j f(x)$ exists and is continuous for $|x-y| < \varepsilon$, and we wish to show that $Df(y)$ exists. $(y \in A)$.

Consider $h \in \mathbb{R}^m$ with $0 < |h| < \varepsilon$. $(h = (h_1, \dots, h_m))$, and the following sequence.

$P_0 = y$

$P_1 = y + h_1 e_1$

$\vdots$

$P_m = y + h_1 e_1 + \dots + h_m e_m = y + h$

Note that each P_i belongs to the closed cube $C(y; |h|) = C$ centered at y and radius $|h|$.

(i.e. the ball centered at y and radius $|h|$ under the sup norm).

Now,

$$f(y+h) - f(y) = \sum_{j=1}^{m} \left(f(P_j) - f(P_{j-1}) \right) \quad \longrightarrow (*)$$

For a fixed j, we define

$$\phi(t) = f(P_{j-1} + te_j)$$

As t varies over $[0, h_j]$, $P_{j-1} + te_j$ ranges from P_{j-1} to P_j

Note that this range lies in C

$\Rightarrow$ ϕ is defined on an open interval containing $[0, h_j]$.

As t varies, since only the jth component $P_{j-1} + te_j$ varies, it follows that $D_j f$ exists at each point of A.

$\Rightarrow$ ϕ is defined for all t and is differentiable in an open interval about $[0, h_j]$.

$\Rightarrow$ By the Mean Value Theorem, we have: (ϕ is cont and diff in $[0, h_j]$)

$$\phi(h_j) - \phi(0) = \phi'(c_j)h_j, \text{ where } c_j \in (0, h_j)$$

$$\Rightarrow f(P_j) - f(P_{j-1}) = D_j f(q_j) h_j, \quad (**)$$

where $q_j = P_{j-1} + c_j e_j$ (Note that this lies in the line segment joining P_{j-1} to P_j) $\subset C$.

Futhermore, $(**)$ holds for $h_j \neq 0$, and also trivially for $h_j = 0$, for any $q_j \in C$.

By applying (**), we rewrite (*), to get:

$$f(y+h) - f(y) = \underline{\underline{\sum_{j=1}^{n} D_j f(q_j) h_j}}, \quad \hookrightarrow (***)$$

for each $q_j \in C$

Now, let $B = [D_1 f(y), \ldots, D_m f(y)]$

Then

$$B.h = \sum_{j=1}^{m} D_j f(y) h_j$$

Using (***), we have

$$\frac{f(y+h) - f(y) - B.h}{|h|} = \sum_{j=1}^{m} \frac{[D_j f(q_j) - D_j f(y)] h_j}{|h|}$$

Allow $h \to 0$, we have

$x_j \to y$ (Since C is centered at y)

$$\Rightarrow \lim_{h \to 0} \frac{f(y+h) - f(y) - B.h}{|h|} \to 0 \quad \blacksquare$$

with $B = Df(y)$.

Theorem. Let $f: A(\subset \mathbb{R}^n) \to \mathbb{R}$ be a C^2 function. Then for each $x \in X$, we have:

$$D_k D_j f(x) = D_j D_k f(x)$$

Proof.

Since the partial derivative is computed by letting all other variables other than

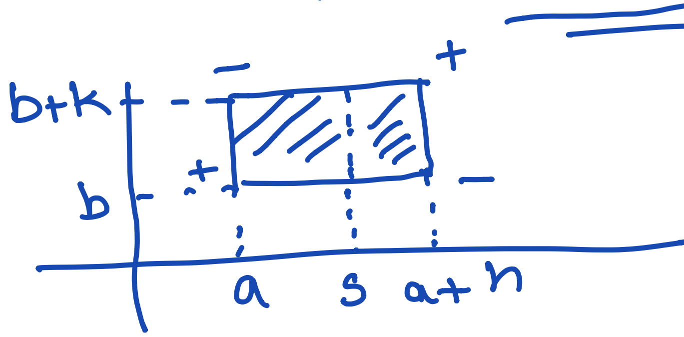
x_k and x_j to remain constant, it suffices to consider the case when $n=2$.

So let $f : \overset{\subset \mathbb{R}^2}{A} \to \mathbb{R}$ be C^2.

Let $Q = [a, a+h] \times [b, b+k]$ be a rectangle in A.

Define

$$\lambda(h,k) = f(a,b) - f(a+h, b) - f(a, b+k) + f(a+h, b+k)$$



Claim. $\exists$ points $p, q \in Q$ such that

(i) $\lambda(h,k) = D_2 D_1 f(p) \cdot hk$

(ii) $\lambda(h,k) = D_1 D_2 f(q) \cdot hk$

It suffices to prove the first assertion (i), as the second would then follow by symmetry.

Let

$$\phi(s) = f(s, b+k) - f(s, b)$$

Then $\phi(a+h) - \phi(a) = \lambda(h,k)$

Since $D_1 f$ exists in A,

ϕ is differentiable in an open interval containing $[a, a+h]$

By the MVT,

$$\phi(a+h) - \phi(a) = \phi'(s_0)\,h, \text{ for some } s_0 \in (a, a+h). \quad (1)$$

$$(1) \implies \lambda(h,k) = \left[D_1 f(s_0, b+k) - D_1 f(s_0, b)\right] \cdot h$$

Now consider $D_1 f(s_0, t)$. Since $D_2 D_1 f$ exists in A, it is differentiable for t in an open interval about $[b, b+k]$.

By applying MVT, we have:

$$D_1 f(s_0, b+k) - D_1 f(s_0, b) = D_2 D_1 f(s_0, t_0) \cdot k$$

for some $t_0 \in (b, b+k)$, which proves our claim.

Now let $\underline{x = (a,b)} \in A$, and for $t > 0$, let

$$\underline{Q_t = [a, a+t] \times [b, b+t]}.$$

By our Claim, for sufficiently small t, we have $Q_t \subset A$, and so we have:

$$\underline{\lambda(t,t) = D_2 D_1 f(P_t) \cdot t^2}, \text{ for some } \underline{P_t \in Q_t}.$$

Letting $t \to 0$, we see that

$$P_t \longrightarrow x$$

Since $D_2 D_1 f$ is continuous, we have:

$$\underline{\underline{\frac{\lambda(t,t)}{t^2}}} \longrightarrow D_2 D_1 f(x)$$

as $\underline{t \to 0}$

Similarly

$$\frac{\lambda(t,t)}{t^2} \longrightarrow D_1 D_2 f(x)$$

as $t \to 0$

$\blacksquare$

Chain rule

Theorem. Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$, and let $f : A \longrightarrow \mathbb{R}^n$, $g : B \longrightarrow \mathbb{R}^p$ with $f(A) \subset B$. Suppose that $f(a) = b$. If f is diff at $\underline{\underline{a}}$, and g is diff at $\underline{\underline{b}}$, then $g \circ f$

is diff at a. Furthermore,

$$\underbrace{D(g\circ f)(a)}_{(p\times m)} = \underbrace{Dg(b)}_{(p\times n)}\cdot\underbrace{Df(a)}_{(n\times m)}$$

<u>Proof.</u>

Let $\overset{\text{arbitrary}}{x} \in \mathbb{R}^m$ and $\overset{\text{arbitrary}}{y} \in \mathbb{R}^n$.

We choose ε so that $g(y)$ is well-defined on $|y-b|<\varepsilon$, and we choose δ so that $|f(x)-b|<\varepsilon$, whenever $|x-a|<\delta$ (due to the cont. of f).

Let $\underline{\Delta(h) = f(a+h) - f(a)}$, which is defined for $|h|<\delta$.

<u>Claim</u>. $\dfrac{|\Delta h|}{|h|}$ is bounded for h in some deleted nbhd of 0.

Define

$$F(h) = \begin{cases} \dfrac{\Delta(h) - Df(a)\cdot h}{|h|}, & 0<|h|<\delta \\ 0, & \text{otherwise.} \end{cases}$$

Note that since f is diff. at a, we have F is cont. at 0. $\left(\lim_{h\to 0} F(h) = F(0)\right)$

Furthermore,

$$\Delta(h) = Df(a)\cdot h + |h|\, F(h) \quad (*)$$

for $0<|h|<\delta$ (and $h=0$).

$$|\Delta(h)| = |Df(a)||h| + |h||F(h)|$$

(Cauchy-Schwarz and Triangle)

$$\leq \underline{\underline{m}}|Df(a)||h| + |h||F(h)| \quad \text{① }$$

Since $|F(h)|$ is bounded for h in a nbhd of 0, so is

$$\frac{|\Delta(h)|}{|h|} \leq m|Df(a)| + |F(h)| \quad \left(\frac{①}{|h|}\right)$$

$\rightarrow \dfrac{|\Delta(h)|}{|h|}$ is bounded ∎ (Proves claim)

Now, we repeat claim for g, by defining

$$G(k) = \begin{cases} \dfrac{g(b+k)-g(b)-Dg(b)\cdot k}{|k|}, & 0<|k|<\varepsilon \\ 0, & \text{otherwise} \end{cases}$$

For $|k|<\varepsilon$, G satisfies

$$g(b+k)-g(b)=Dg(b)\cdot k+|k|G(k) \quad \text{(**)}$$

Now let $h\in\mathbb{R}^m$ with $|h|<\delta$.

Then $|\Delta(h)|<\varepsilon$, so we may substitute $\Delta(h)$ for k in (**),

(Note that

$$b+k=b+\Delta(h)=f(a)+\Delta h$$
$$=f(a+h))$$

we have:

$$g\circ f(a+h)-g\circ f(a)=Dg(b)\Delta h$$
$$+|\Delta(h)|\,G(\Delta(h)) \quad \text{(***)}$$

Rewriting (***) using (*)

$$\frac{1}{|h|}\left[g\circ f(a+h) - g\circ f(a) - Dg(b).Df(a)h\right]$$

$$= Dg(b)F(h) + \frac{1}{|h|}|\Delta(h)|G(\Delta(h)),$$

which holds for $0<|h|<\delta$.

Since as $h\to 0$, $F(h)\longrightarrow 0$, $G(\Delta(h))\to 0$,

and $\frac{|\Delta(h)|}{|h|}$ is bounded (claim),

we have:

$$\lim_{h\to 0}\left[g\circ f(a+h) - g\circ f(a) - Dg(b)\cdot Df(a)h\right] = 0$$

$$\Rightarrow D(g\circ f)(a) = Dg(b)\cdot Df(a)$$

(by uniqueness of limit) ■

Corollary. Let A be open in $\mathbb{R}^m$, and B be open in $\mathbb{R}^n$. Let $f: A \longrightarrow \mathbb{R}^m$ and $g: B \longrightarrow \mathbb{R}^p$ with $f(A) \subset B$. If f, g are of class C^r, then so is $g \circ f$.

Proof. Case $r=1$, i.e f, g are of class C^1.

Then Dg has continuous real-valued components on B.

f is cont. on A

$\implies Dg \circ f(x)$ also has cont

components on each $x \in A$

$$\left(Dg\circ f(x) \underset{\text{chain rule}}{=} Dg(f(x))\, Df(x) \right)$$

$\Rightarrow$ $g\circ f$ is of class C^1 on A.

Now use induction on r to complete the proof (Exercise) ∎

<u>Theorem</u> (Mean Value Theorem). Let A be open in $\mathbb{R}^m$, and let $f : A \to \mathbb{R}$ be diff on A. If A contains the line segments with end

points a and $a+h$, for some $a \in A$ then

$\exists\ c = a + t_0 h$, $0 < t_0 < 1$,

such that

$$f(a+h) - f(a) = Df(c) \cdot h.$$

<u>Proof</u>.

Set $\phi(t) = f(a+th)$.

Then ϕ is defined for all t in an open interval about $[0,1]$, and is also diff. as

$$\phi'(t) = Df(a+th) \cdot h$$

By the MVT (dim 1), we have

$$\phi(1) - \phi(0) = \phi'(t_0) \cdot 1$$

$$t_0 \in (0,1)$$

$$\Longrightarrow \quad f(a+h) - f(a) = Df(a+t_0h) \cdot h$$

Theorem. Let $A \subset \mathbb{R}^n$ be open, and let $f : A \longrightarrow \mathbb{R}^n$ with $f(a) = b$. Suppose that g maps a nbhd of b into $\mathbb{R}^n$ such that $g(b) = a$ and $(g \circ f)(x) = x$ $\forall$ x in that nbhd (of a). If f is diff at a and if g is diff at b, then

$$Dg(b) = [Df(a)]^{-1}.$$

Proof. We are given that $(g \circ f)(x) = id(x)$, $\forall$ x in some nbhd of a.

By Chain rule, we have

$$Dg(b)\,Df(a) = I_n$$

$\hookrightarrow$ $n \times n$ identity matrix

$$\Rightarrow Dg(b) = Df(a)^{-1} \blacksquare$$

Inverse Function Theorem

Lemma 8.1. Let A be open in $\mathbb{R}^n$, and let $f: A \longrightarrow \mathbb{R}^n$ be of class C^1. If $Df(x)$

is non-singular, then $\exists$ an $\underline{\underline{\alpha, \varepsilon > 0}}$ such that

$$|f(x_0) - f(x_1)| \geq \alpha |x_0 - x_1|,$$

for all $x_0, x_1 \in C(x; \varepsilon)$.

<u>Proof.</u>

Let $\underline{E = Df(x)}$. By assumption E is non-singular.

$$|x_0 - x_1| = |E^{-1}(E \cdot x_0 - E \cdot x_1)|$$

$$\leq n|E^{-1}||E \cdot x_0 - E x_1|$$

If we set $2\alpha = \dfrac{1}{n|E^{-1}|}$, then for all $x_0, x_1 \in \mathbb{R}^n$

$$|Ex_0 - Ex_1| \geq 2\alpha |x_0 - x_1|$$

Let $\underline{H(y) = f(y) - E\cdot y}$

Then $DH(y) = Df(y) - E$

$$\Rightarrow DH(x) = 0 \quad (\underset{\underset{E}{\shortparallel}}{Df(x)} - E)$$

$\because$ H is C^1, we choose $\varepsilon > 0$ such that

$$|DH(y)| < \frac{\alpha}{n}, \text{ for}$$

$y \in \underline{C = C(x;\varepsilon)}$.

By MVT applied to the ith component H_i (of H), we get:

$$|H_i(x_0) - H_i(x_1)| = |\underline{\underline{DH_i(c)}}(x_0 - x_1)|$$
$$\leq n \frac{\alpha}{n} |x_0 - x_1|$$
$$\forall \quad x_0, x_1 \in C \qquad L(*)$$

Then for $x_0, x_1 \in C$, we have (by $(*)$)

$$\alpha|x_0 - x_1| \geq |H(x_0) - H(x_1)|$$
$$\geq |E x_1 - E x_0|$$
$$- |f(x_1) - f(x_0)|$$
(triangle inequality)
$$\geq 2\alpha|x_1 - x_0|$$
$$- |f(x_1) - f(x_0)|$$

∎

Theorem. Let A be open in $\mathbb{R}^n$, and $f : A \to \mathbb{R}^n$ be of class C^r with $B = f(A)$. If f is injective on A and if $Df(x)$ is non-singular for $x \in A$, then the set B is open in $\mathbb{R}^n$ and the inverse $g : B \to A$ is of class C^r.

Theorem (Inverse Function Theorem). Let A be open in $\mathbb{R}^n$, and $f : A \to \mathbb{R}^n$ be of class C^r. If $Df(x)$ is non-singular at the point $y \in A$, $\exists$ a nbhd $U \ni y$ such that $f|_U : U \longrightarrow f(U)\ (= V \subset \mathbb{R}^n)$ is injective and the inverse g is of class C^r.

$(g : f(U) \longrightarrow U)$

<u>Proof</u>. By the Lemma, $\exists$ a nbhd $\underline{\underline{U_0}}$ of y on which f is injective.

Since $\det(Df(x))$ is a continuous function of x

and $\det Df(y) \neq 0$, $\exists$ a nbhd $\underline{\underline{U_1}} \ni y$ such that $\det Df(x) \neq 0$ on U_1.

If $U = U_0 \cap U_1$, then the hypothesis of the previous theorem is satisfied for $f : U \longrightarrow \mathbb{R}^n$, and thus the IVT follows $\blacksquare$

Implicit Function Theorem

Suppose that the equation $f(x,y)=0$ determines y as a diff. function of x (i.e $y=g(x)$ say). Then we have:

$$f(x, g(x)) = 0$$

$$\frac{\partial f}{\partial x} + \left(\frac{\partial f}{\partial y}\right) g'(x) = 0$$

$$\Rightarrow \quad g'(x) = \frac{-\partial f/\partial x}{\partial f/\partial y}$$

Provided that $\frac{\partial f}{\partial y} \neq 0$ at $(x, g(x))$ —— $(*)$

The last condition $(*)$ is in fact sufficient.

<u>i.e.</u>: If $f(x,y)$ has the property $\frac{\partial f}{\partial y} \neq 0$ at (a,b) that is also a solution to $f(x,y)=0$ $(f(a,b)=0)$, then this equation determines y has a function of x near a.

In general, suppose that $f: \mathbb{R}^{k+n} \longrightarrow \mathbb{R}^n$ is of class C^1. Then $f(x_1, \dots, x_{k+n}) = 0$ is equivalent to a system of n scalars in $k+n$ unknowns.

<u>Notation</u>. Let $f: A (\subset \mathbb{R}^n) \longrightarrow \mathbb{R}^m$ be diff. Let f have component functions f_i, for $1 \le i \le n$.

Then

$$Df = \boxed{\frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots x_m)}}$$

$$= \frac{\partial f}{\partial X} \quad (X = (x_1, \dots x_m))$$

<u>Theorem</u>. Let $A \subset \mathbb{R}^{k+n}$ be open, and let $f : A \to \mathbb{R}^n$ be diff. View f as

$f = f(X, Y)$, $X \in \mathbb{R}^k$ and $Y \in \mathbb{R}^n$.

Then Df has the form

$$Df = \left[\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}\right].$$

Suppose that $\exists$ a diff. function

$g : \underbrace{B}_{k} \xrightarrow{\text{open}} \underbrace{\mathbb{R}^n}_{n}$ such that

$f(x, g(x)) = 0$, for $x \in B$.

Then

$$Dg(x) = -\left[\frac{\partial f}{\partial y}(x, g(x))\right]^{-1} \cdot \frac{\partial f}{\partial x}(x, g(x))$$

Proof. Define $h : B \to \mathbb{R}^{k+n}$

$$h(x) = (\underbrace{x}_{k}, \underbrace{g(x)}_{n})$$

Then by our hypothesis the function,

$$H(x) = f(h(x))$$
$$= f(x, g(x))$$

is defined and equals 0, $\forall\ x \in B$.

By chain rule, we have:

$$\underline{\underline{0}} = DH(x) = Df(h(x)) \circ Dh(x)$$

$$= \left[\frac{\partial f}{\partial x}(h(x)) \quad \frac{\partial f}{\partial y}(h(x)) \right] \circ \begin{bmatrix} I_K \\ Dg(x) \end{bmatrix}$$

$$(\forall\ x \in B)$$

$$\Rightarrow 0 = \left(\frac{\partial f}{\partial x}\right)(h(x)) + \frac{\partial f}{\partial y}(h(x)) \cdot Dg(x),$$

from which our assertion follows ∎

<u>Note</u>. In other words, $\boxed{\frac{\partial f}{\partial y}}$ must be nonsingular to compute Dg.

We will now prove that it suffices to guarantee existence of g.

Theorem. (Implicit function theorem)

Let $f : S (\subset \mathbb{R}^{k+n}) \longrightarrow \mathbb{R}^n$ be of class C^r. Write $f = f(x, y)$, for $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$. Let $(A, B) \in S$ such that $f(A, B) = 0$ and $\det \frac{\partial f}{\partial y}(A, B) \neq 0$

Then $\exists$ a nbhd W of A in $\mathbb{R}^k$ and a unique continuous function $g : W \to \mathbb{R}^n$ such that $g(A) = B$ and $f(x, g(x)) = 0$, $\forall\ x \in W$.

Moreover, g is of class C^r.

Proof. Define.

$$\underline{F(x,y) = (x, f(x,y))}$$

Then $F: A (\subset \mathbb{R}^{k+m}) \longrightarrow \mathbb{R}^{k+n}$

and

$$DF = \begin{bmatrix} I_k & 0 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$$

Then

$$\det(DF) = \underline{\underline{\det\left(\frac{\partial f}{\partial y}\right)}} \neq 0$$

(at (A,B) by our hypothesis)

($\Rightarrow$ DF is non-singular at $\underline{\underline{(A,B)}}$)

Applying IFT (Inverse F.T) to F, we get an open set $U \times V \ (\subset \mathbb{R}^{k+n})$ and $U \times V \ni (A, B)$ such that:

(a) $F|_{U \times V} : U \times V \to F(U \times V)$ is injective, where $F(U \times V)$ $(\ni A, 0)$ is open.

(b) The inverse $G: F(U \times V) \to U \times V$ exists and is of class C^r.

Since $\underline{F(x, y) = (x, f(x, y))}$, we have $\underline{\underline{(x,y) = G}}\,(x, f(x,y))$

So, G preserves X (ie first k coordinates)

$$\Rightarrow \underline{G(x,z) = (x, \underline{h}(x,z))}$$

$x \in \mathbb{R}^k$, $z \in \mathbb{R}^n$, and h is a C^r function mapping

$$\underline{f(U \times V) \xrightarrow{G} \mathbb{R}^n.}$$

Now let $W \supset A$ be a connected nbhd in $\mathbb{R}^k$ chosen small enough

$\mathbb{R}^n$

so that $W \times \underline{\underline{0}}^{\epsilon} \subset f(U \times V)$.

If $X \in W$, then $(X, 0) \subset f(U \times V)$.

so $G(X, 0) = (X, h(X, 0))$

$\Rightarrow (X, 0) = F(X, h(X, 0))$

$= (X, f(X, h(X, 0)))$

$\Rightarrow \underline{\underline{0}} = \underline{f(X, h(X, 0))}$ (1)

Let $g(X) = h(X, 0)$, $\forall X \in W$

Then g satisfies

$f(X, g(X)) = 0 \rightarrow$ from (1)

Moreover,

$$(A, \underline{\underline{B}}) = G(A, 0)$$
$$= (A, \underline{\underline{h(A,0)}})$$
$$\Rightarrow B = h(A,0) = g(A),$$

as desired. Moreover g is of class C^r. (Inverse F.T)

Uniqueness of g

Suppose that $g_0 : W \to \mathbb{R}^n$ is another continuous function satisfying the conclusion of the theorem.

Then

$$g(A) = g_0(A)$$

Now suppose that for some $A_0 \in W$.

$$g(A_0) = g_0(A_0),$$

Then $\exists$ a nbhd $W_0 \ni A_0$ such that $g_0(W_0) \subset W$

Since $f(X, g_0(X)) = 0$, for $X \in W_0$, we have:

$$F(X, g_0(X)) = (X, 0), \text{ so}$$

$$\begin{aligned}(X, g_0(X)) &= G(X, 0) \\ &= (X, h(X, 0)) \\ &\forall\, X \in W_0\end{aligned}$$

$$\Rightarrow \quad \underline{g_0 = g \text{ on } W_0}$$

Finally, we consider

$$\{x \in W \mid |g(x) - g_0(x)| = 0\}$$

This is open (from what we just showed)

Moreover,

$$\{x \in W \mid |g(x) - g_0(x)| > 0\} \text{ is open.}$$

$$\| \quad |g - g_0|^{-1}(0, \infty)$$

Since W is connected, we have

$$\underline{\{x \in W \mid |g(x) - g_0(x)| = 0\}}$$ is both open and closed,

and so we have

$$\{x \in W \mid |g(x) - g_0(x)| = 0\} = W$$

$$\Rightarrow g = g_0 \text{ on } W \quad \blacksquare$$

Examples

(a) $f : U (\subset \mathbb{R}^5) \longrightarrow \mathbb{R}^2$ is of class C^r

Assuming $f(x, y, z, u, v) = 0$, we wish to solve y, u in terms of x, z, v.

By the Implicit Func. Thm,

if $A \in U$ such that

$f(A) = 0$ and $\det \frac{\partial f}{\partial(y,u)}(A) \neq 0$,

then $y = \phi(x,z,v)$, $u = \psi(x,z,v)$

Furthermore,

$$\frac{\partial(\phi,\psi)}{\partial(x,z,v)} = -\left[\frac{\partial f}{\partial(y,u)}\right]^{-1}\left[\frac{\partial f}{\partial(x,z,v)}\right]$$

(b) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$f(x,y) = x^2 + y^2 - 5$$

Note that $f(1,2) = 0$,

and $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \neq 0$ at $(1,2)$

By the ImFT, we can solved y in terms of x

In particular

$$\underline{y = g(x) = \overset{\text{continuous}}{\sqrt{5-x^2}}}$$

in a nbhd of $(1,2)$

What about

$$h(x) = \begin{cases} \sqrt{5-x^2}, & x \geq 1 \\ -\sqrt{5-x^2}, & x < 1 \end{cases}$$

(Not continuous)

(c) Consider the same example in (b) above in the nbhd of $(\sqrt{5},0)$

Im function theorem is not applicable.

$\frac{\partial f}{\partial y}(\sqrt{5},0) = 0$ (even though $f(\sqrt{5},0) = 0$)

It turns out that it does not have an implicit solution.

(d) $f : \mathbb{R}^2 \longrightarrow \mathbb{R} : (x,y) \mapsto x^2 - y^3$

$f(0,0) = 0$ and $\frac{\partial f}{\partial y}(0,0) = 0$

$\Longrightarrow$ Im F.T is not applicable.

$\not\Rightarrow$ No solution of y in terms of x

$\left(y = x^{2/3} \right)$

Integration

Defn. Let $Q = [a_1 \times b_1] \times \cdots \times [a_n \times b_n]$
$$= \prod_{i=1}^{n} [a_i \times b_i]$$
be a rectangle in $\mathbb{R}^n$. Then:

(i) $[a_i, b_i]$ is called a component interval of Q.

(ii) $\max\{|b_i - a_i|\}$ is called the width of Q.

(iii) $v(Q) = (b_1 - a_1)(b_2 - a_2) \ldots (b_n - a_n)$ is called the volume of Q.

Note. When $n=1$, $v(Q) =$ width of Q = length of $[a,b]$.

__Defn__. Given an interval $[a,b] \subset \mathbb{R}$, a partition of $[a,b]$ is a collection $P = \{t_0, \dots, t_n\}$ of points in $[a,b]$ such that:

$$a = t_0 < t_1 < \dots < t_k = b$$

Each of the intervals $[t_{i-1}, t_i]$, $1 \le i \le k$ is called a __subinterval determined by P__.

__Defn__. Given a rectangle $Q = \prod_{i=1}^{n} [a_i, b_i]$ in $\mathbb{R}^n$, a __partition P of Q__ is an n-tuple __$(P_1, \dots, P_n)$__ such that

P_j is a partition of $[a_j, b_j]$, for each j. If for each j, I_j is a subinterval determined by P_j (of $[a_j, b_j]$), the

$$R = \prod_{i=1}^{n} I_j \ (= I_1 \times \cdots \times I_n)$$

is called a <u>subrectangle</u> determined by P of the rectangle Q.

The max width of these subrectangles is called a <u>mesh</u> of P.

Defn. Let $Q \subset \mathbb{R}^n$ be a rectangle, and let $f : Q \to \mathbb{R}$ be a bounded function.

$(|f(x)| \leq M > 0, \ \forall \ x \in Q)$

Let P be a partition of Q. For each subrectangle R determined by P, let:

$$m_R(f) = \{\inf(f(x)) : x \in R\}$$

$$M_R(f) = \{\sup(f(x)) : x \in R\}$$

We define lower sum and upper sum (resp.) of f, determined by P by

$$\underline{\underline{L(f,P)}} = \sum_R m_R(f) \cdot \underline{\underline{v(R)}}$$

$$\underline{\underline{U(f,P)}} = \sum_R M_R(f) \cdot v(R)$$

Defn.

Let $P = (P_1, \ldots, P_n)$ be a partition of Q. If P'' is a partition of Q obtained from P by adjoining some additional points to some or all of the partitions $P_1, \ldots, P_n$, then P'' is called a <u>refinement</u> <u>of P</u>.

__Defn__. Given partitions $\mathcal{P} = (P_1, \dots, P_n)$ and $\mathcal{P}' = (P_1', \dots, P_n')$ of Q, the partition

$$\mathcal{P}'' = (P_1 \cup P_1', \dots, P_n \cup P_n')$$

is called a __common refinement__ of $\mathcal{P}$ and $\mathcal{P}'$.

(Note: P_i and P_i' can have a non-trivial intersection)

__Lemma__. Let $\mathcal{P}$ be a partition of Q, and let $f : Q \longrightarrow \mathbb{R}$ be a bounded function. If $\mathcal{P}''$ is a refinement of $\mathcal{P}$, then:

$\underline{L(f,P) \leq L(f,P'')}$ and $U(f,P'')$
$\underline{\leq U(f,P)}$.

<u>Proof</u>.

Let $Q = \overbrace{[a_1, b_1]}^{P_1} \times \cdots \times [a_n, b_n]$

It suffices to prove the lemma for the case of a refinement P'' obtained by adjoining a single point of $P = (P_1, \ldots, P_n)$

WLOG, we may assume P'' is obtained by adding q to P_1:

$a_1 = t_0 < t_1 < \cdots < t_k = b_1$ and

$q \in (t_{i-1}, t_i)$.

Most subrectangles determined by P are also subrectangles of P'', except subrectangles of the form:

$$\underline{R_S = [t_{i-1}, t_i] \times S}, \text{ where}$$

S is a subrectangle of $[a_2, b_2] \times \ldots \times [a_n, b_n]$.

In P'', R_S would be replaced by:

$$R_S' = [t_{i-1}, q] \times S \quad \text{and}$$
$$R_S'' = [q, t_i] \times S.$$

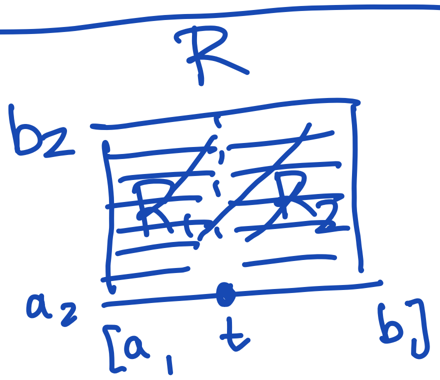
Clearly,

$$m_{R_s}(f) \leq m_{R_s'}(f) \quad (R_s' \subset R_s)$$

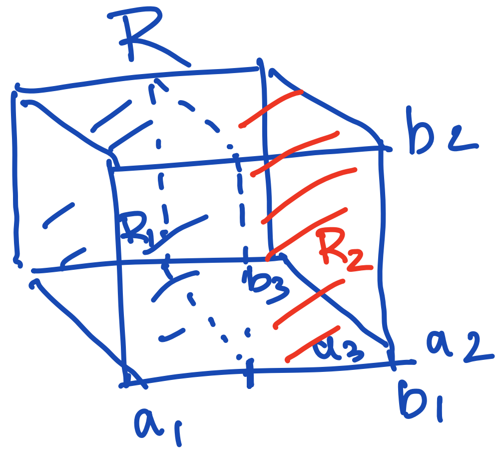
and $$m_{R_s}(f) \leq m_{R_s''}(f) \quad (R_s'' \subset R_s)$$

Since $\nu(R_s) = \nu(R_s') + \nu(R_s'')$, by direct computation, we get:

$$m_{R_s}(f)\,\nu(R_s) \leq m_{R_s'}(f)\,\nu(R_s') + m_{R_s''}(f)\,\nu(R_s'') \quad \text{—}(*)$$



$$\nu(R) = \nu(R_1) + \nu(R_2)$$



$$v(R) = v(R_1) + v(R_2)$$

Since (*) holds for every R_S, we have:

$$L(f,P) \leq L(f,P'')$$

Similarly, $U(f,P) \geq U(f,P'')$

$\blacksquare$

Lemma. Let Q be a rectangle, and let $f: Q \to \mathbb{R}$ be bounded. If P' and P are any two partitions of Q, then:

$$L(f,P) \leq U(f,P').$$

Proof. When $P = P'$, it is follows immediately.

Otherwise, consider the common refinement $P'' = P \cup P'$. Then, we have:

$$L(f,P) \leq L(f,P'') \leq U(f,P'') \leq U(f,P')$$

∎

Defn. Let Q be a rectangle and let $f: Q \to \mathbb{R}$ be a bounded function. Then we define:

$$\underline{\int_Q} f := \sup_P \{L(f,P)\}$$

$$\overline{\int_Q} f := \inf_P \{U(f,P)\},$$

where P ranges over all partitions of Q.

$\underline{\int_Q} f$ and $\overline{\int_Q} f$ are called the <u>lower</u> and <u>upper integrals</u> (resp.) of f over Q.

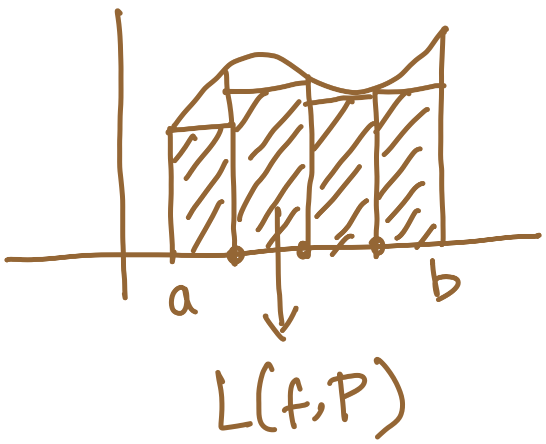
Defn. If $\underline{\int_Q} f = \overline{\int_Q} f$, then f is said to be <u>integrable over Q</u>, and the common value is called the <u>integral of f over Q</u>.

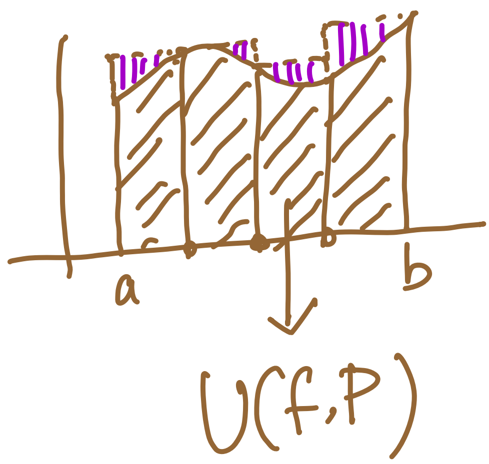
<u>Example</u> (a) $f : [a,b] \to \mathbb{R}$ be a non-negative bounded function

If P is a partition of $[a,b]$, then:

$L(f,P) =$ Total area of rectangles inscribed between f and x-axis. (inner area)

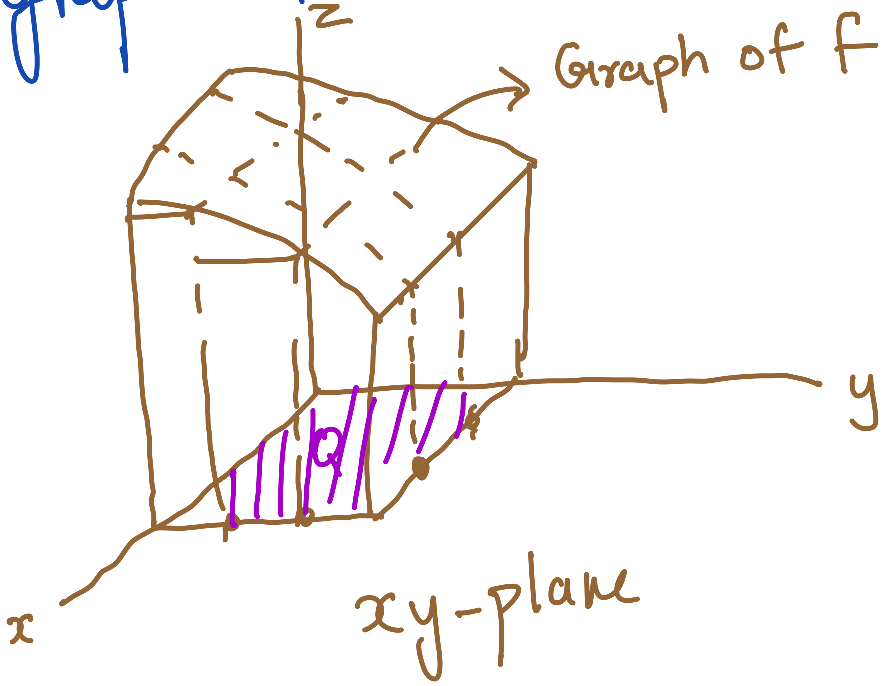
$U(f,P) =$ Total area of rectangles circumscribed about the region between f and x-axis.





(b) Similarly, if $Q \subset \mathbb{R}^2$ and $f: Q \to \mathbb{R}$ is non-negative and bounded, then:

we can visualize $L(f,P)$ (resp. $U(f,P)$) to be the total volume inscribed (resp. circumscribed) in the region between the graph of f and the xy-plane.



Example. $I = [0,1]$, and let $f : I \longrightarrow \mathbb{R}$ be defined By:

$$f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q} \\ 1, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

(Popcorn function or Dirichlet function)

If P is a partition of I, and R is a subinterval of P, then we have :

$$m_R(f) = 0$$

$$M_R(f) = 1$$

(since R contains both rational and irrational numbers).

Therefore,

$$L(f,P) = \sum_R 0 \cdot \gamma(R) = 0$$

$$U(f,P) = \sum_R 1 \cdot \gamma(R) = 1$$

$$\Rightarrow \quad \underline{\int_R} f = 0 \quad \text{and} \quad \overline{\int_R} f = 1$$

$\Rightarrow$ f is not integrable.

<u>Theorem</u> Let Q be a rectangle, and let $f : Q \to \mathbb{R}$ be a bounded function. Then:

$$\underline{\int_Q} f \leq \overline{\int_Q} f .$$

Moreover, <u>equality holds</u> iff given $\varepsilon > 0$, $\exists$ a partition P of Q such that:

$$U(f,P) - L(f,P) < \varepsilon .$$

(Riemann Condition)

Proof. Let P' be a partition of Q.

Then $L(f,P) \leq U(f,P')$, for every partition P of Q.

$$\Rightarrow \underline{\int_Q} f \leq U(f,P')$$

Since P' is arbitrary, we have

$$\underline{\int_Q} f \leq \overline{\int_Q} f$$

This concludes the proof of the first part.

For the second part of the assertion, assume first that

$$\underline{\int_Q} f = \overline{\int}^Q f \quad (\Rightarrow)$$

Choose P, P' such that

$$\left.\begin{aligned} &\underline{\int_Q} f - L(f,P) < \frac{\varepsilon}{2} \\ &\text{and} \\ &U(f,P') - \overline{\int}^Q f < \frac{\varepsilon}{2} \end{aligned}\right\} (*)$$

If $P'' = P \cup P'$ (i.e. the common refinement), then we have :

$$L(f,P) \leq \underline{L(f,P'')} \leq \int_Q f$$

$$\leq U(f,P'')$$

$$\leq U(f,P')$$

$$\Longrightarrow U(f,P'') - L(f,P'') < \varepsilon$$

(Check!)

(from (*))

$(\Longleftarrow)$ If $\varepsilon = \overline{\int_Q} f - \underline{\int_Q} f > 0$

(not-integrable)

For any partition P of Q, we have:

$$L(f,P) \leq \underline{\int_Q} f < \overline{\int_Q} f \leq U(f,P)$$

$$\Rightarrow \quad U(f,P) - L(f,P) \geq \overline{\int_Q} f - \underline{\int_Q} f = \varepsilon \quad \blacksquare$$

Theorem

Every constant function ($f(x) = c$) is integrable. Indeed, if Q is a rectangle and if P is a partition of Q, then

$$\int_Q c = c \cdot v(Q) = c \sum_R v(R),$$

where the (last) summation extends over all subrectangles determined by P.

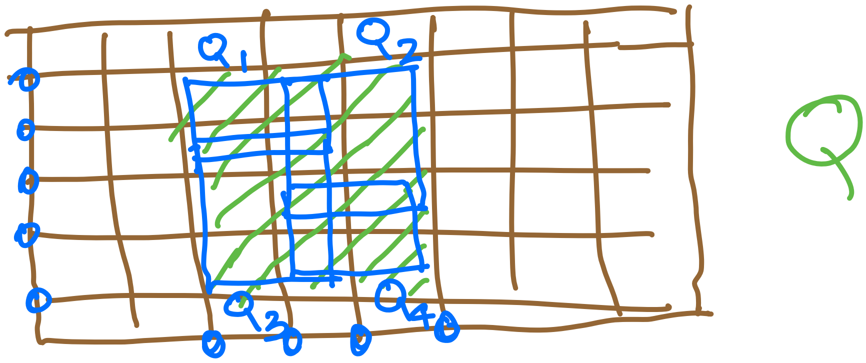
Corollary. Let Q be a rectangle in $\mathbb{R}^n$, and let $\{Q_1, \dots, Q_k\}$ be a finite collection of rectangles that cover Q. Then:

$$\nu(Q) \leq \sum_{i=1}^{k} \nu(Q_i).$$

Proof.

Choose a rectangle Q' containing all rectangles $Q_1, \dots, Q_k$

Use the end points of intervals of $Q, Q_1, \dots, Q_k$ to define a partition P of Q'.



Then each of $Q, Q_1, \ldots, Q_K$ is a union of subrectangles determined by P.

By the preceding theorem, we have:

$$\gamma(Q) = \sum_{R \subset Q} \gamma(R)$$

As each subrectangle R (of Q) is contained in at least one of $Q_1, \ldots, Q_K$, we have:

$$\sum_{R \subset Q} v(R) \leq \sum_{i=1}^{K} \left(\sum_{R \subset Q_i} v(R) \right)$$

$$= \sum_{i=1}^{K} v(Q_i).$$

Existence of the integral

<u>Defn</u>. A subset $A \subset \mathbb{R}^n$ is said to be of <u>measure zero</u> in $\mathbb{R}^n$ if for every $\varepsilon > 0$, $\exists$ a covering $Q_1, Q_2, \ldots$ of A by countably many rectangles such that:

$$\sum_{i=1}^{\infty} v(Q_i) < \varepsilon.$$

Theorem.

(a) If $B \subset A$ and A is of measure zero in $\mathbb{R}^n$, then so does B.

(b) Let $A = \bigcup_{j=1}^{\infty} A_j$, where each A_j has measure zero. Then A has measure zero.

(c) A set $A \subset \mathbb{R}^n$ has measure zero iff for every $\varepsilon > 0$, $\exists$ a countable covering of A by open rectangles $\underline{\underline{Q_1^\circ}}, \underline{\underline{Q_2^\circ}}, \ldots$ such that

interiors of rectangles

$$\sum_{i=1}^{\infty} v(Q_i) < \varepsilon.$$

(d) If Q is a rectangle in $\mathbb{R}^n$, then $\underline{\underline{\partial Q}}$ has measure 0 in $\mathbb{R}^n$, but Q does not.

<u>Proof</u>

(a) Exercise (obvious).

(b) Given $\varepsilon > 0$, let

$$A_j = \bigcup_{i=1}^{\infty} Q_{ij} \quad \text{such that}$$

$$\sum_{i=1}^{\infty} v(Q_{ij}) < \frac{\varepsilon}{2^j}$$

Then $\{Q_{ij}\}$ is countable and

$$\bigcup_{i,j} Q_{ij} = A \quad \left(A = \bigcup_{j=1}^{\infty} A_j\right)$$

Moreover,

$$\nu(A) \leq \underset{\text{(Previous Thm)}}{\sum_{i,j}} \nu(Q_{ij}) < \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon \quad \blacksquare$$

(c) $(\Longleftarrow)$ If $\{\overset{\circ}{Q_j}\}$ (open rectangles) covers A, then clearly so does $\{Q_j\}$, and so the assertion follows.

$(\Longrightarrow)$ Suppose that A is of measure zero.

Let $A = \bigcup_{j=1}^{\infty} Q_j'$ such that

$$\sum_{j=1}^{\infty} \nu(Q_j') < \frac{\varepsilon}{2}.$$

For each i, choose Q_i such that $Q_i' \subset Q_i^\circ$ and

$\nu(Q_i) \leq 2\nu(Q_i')$.

($\because$ ν is continuous function at the end points of the component intervals)

Think about this!

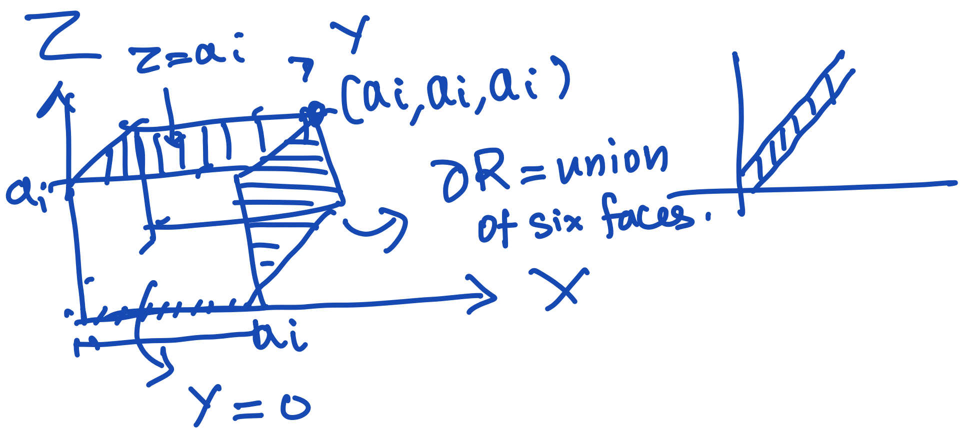
Then $A = \bigcup_{j=1}^{\infty} Q_j^\circ$ and

$$\sum_j \nu(Q_j) < \varepsilon \; c$$

(d) Let $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$

Let an ith face of Q be:

$$F_i = \{ x \in Q \mid x_i = a_i \text{ (or } b_i) \}$$



Note that an ith face has measure zero.

This is because, if $x_i = a_i$, then

$$[a_1,b_1] \times \cdots \times [a_i, a_i+\delta] \times \cdots \times [a_n,b_n]$$

has volume as small as possible (by an appropriate choice of δ).

Then

∂Q = Union of faces in Q, has measure zero.

($\because$ no.. of faces is finite)

Suppose Q has measure zero. Let $\varepsilon = v(Q)$.

Cover Q be open rectangles:

$$Q = \bigcup_{j=1}^{\infty} Q_j^{o} \text{ with } \underline{\underline{\sum_{j=1}^{\infty} v(Q_i) < \varepsilon.}} \ (*)$$

Since Q is compact, cover Q by open sets $Q_1^o, \ldots Q_k^o$ (every open cover has a finite subcover)

But $\displaystyle\sum_{i=1}^{k} v(Q_i) \underset{(\text{from } *)}{<} \varepsilon = v(Q)$ $\#$

<u>Theorem</u>. Let $Q \subset \mathbb{R}^m$, and let $f: Q \to \mathbb{R}$ be bounded. Let $D = \{x \in Q : f \text{ is not cont. at } x\}$

Then $\int_Q f$ exists iff D has measure 0 in $\mathbb{R}^m$.

<u>Example</u>

(a) $f(x) = \begin{cases} 1/x, & x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

Is f integrable in $[-2,2)$.

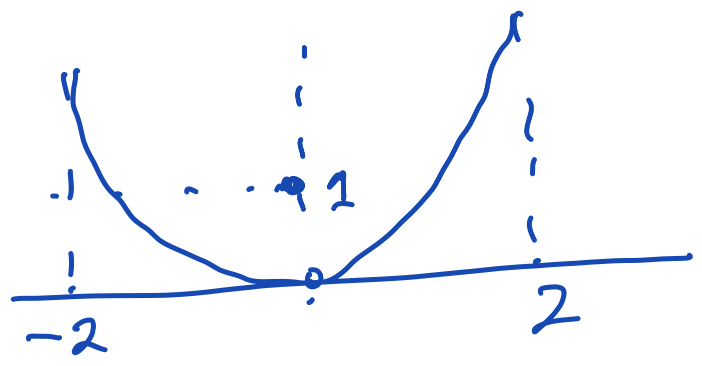
Not conclusive as the

hypothesis of theorem is not satisfied (f is unbounded)

(b) $f(x) = \frac{1}{x^2}, \quad 2 \le x < 4$

This is integrable from the theorem as is f in continuous in $[2, 4)$.

(c) $f(x) = \begin{cases} x^2, & x \in [-2,0) \cup (0,2] \\ 1, & x = 0 \end{cases}$



By theorem, as $\{x : f \text{ is not cont.}\} = \{0\}$, we have that f is integrable.

<u>Theorem</u>. Let Q be a rectangle in $\mathbb{R}^n$ and let $f : Q \to \mathbb{R}$ be integrable.

(a) If f vanishes except on a set of measure zero, then $\int_Q f = 0$

(b) If f is non-negative and $\int_Q f = 0$, then f vanishes (on Q) exept on a set of measure zero.

Proof

(a) Suppose that f vanishes except on a set $E \subset Q$ of measure zero.

Let P be a partition of Q.

If R is a subrectangle of P, then $R \not\subset E$ $(v(R) > 0)$

$\Rightarrow f$ vanishes in some point of R.

Then

$$m_R(f) \leq 0 \quad \text{and} \quad M_R(f) \geq 0$$

$$\Rightarrow L(f, P) \leq 0 \quad \text{and} \quad U(f, P) \geq 0$$

$$\Rightarrow \underline{\int_Q} f \leq 0 \quad \text{and} \quad \overline{\int_Q} f \geq 0$$

$$\Rightarrow \int_Q f = 0 \quad (\text{as } f \text{ is integrable})$$

(b) Suppose that $f(x) \geq 0$ all $x \in Q$, and $\int_Q f = 0$.

Claim. If f is continuous at a, then $f(a) = 0$

(Then by previous theorem, f must vanish except at a set of noncontinuous points of measure zero).

Let f be cont at a, and let $f(a) > 0$. with $\varepsilon = f(a)$.

Then by cont. $\exists\ \delta > 0$

such that $f(x) > \frac{\varepsilon}{2}$ for $|x-a| < \delta$ $(x \in Q)$

Choose a partition P of Q of mesh $< \delta$, and let R_0 be a subrectangle $\ni a$.

Then $m_{R_0}(f) \geq \frac{\varepsilon}{2}$

Moreover, $m_R(f) \geq 0$, for all R.

Therefore, it follows that

$$L(f,P) = \sum_R m_R(f)\, v(R)$$

$$\geq \frac{\varepsilon}{2}\, v(R_0) > 0$$

But $L(f,P) < \underline{\int_Q} f = 0$ #

Evaluation of the integral

Theorem (Fundamental Theorem of Calculus).

(a) If f is continuous on $[a,b]$, and if $F(x) = \int_a^x f$, for $x \in [a,b]$, then $F'(x)$ exists and $F'(x) = f(x)$.

(b) If f is continuous on $[a,b]$, and if g is a function such that $g'(x) = f(x)$

for $x \in [a,b]$, then

$$\int_a^b f = g(b) - g(a).$$

<u>Theorem</u> (Fubini's Theorem). Let $Q = A \times B$, where A is a rectangle in $\mathbb{R}^k$ and B is a rectangle in $\mathbb{R}^n$. Let $f : Q \to \mathbb{R}$ be a bounded function written in the form $f(x,y)$ for $x \in A$ and $y \in B$. For each $x \in A$ consider the integrals

$$\underline{\int}_{y \in B} f(x,y) \quad \text{and} \quad \overline{\int}_{y \in B} f(x,y).$$

If f is integrable over Q, then these two functions
$\left(x \mapsto \underline{\int}_{y \in B} f(x,y) \right)$ and $x \mapsto \overline{\int}_{y \in B} f(x,y))$
of x are integrable over A, and

$$\int_Q f = \int_{x \in A} \underline{\int}_{y \in B} f(x,y) = \int_{x \in A} \overline{\int}_{y \in B} f(x,y)$$

Corollary. Let $Q = A \times B$, where A is a rectangle in $\mathbb{R}^k$ and B is a rectangle in $\mathbb{R}^n$. Let $f: Q \to \mathbb{R}$ be a bounded function. If $\int_Q f$ exists, and if $\int_{y \in B} f(x,y)$ exists for each $x \in A$, then

$$\int_Q f = \int_{x \in A} \int_{y \in B} f(x,y).$$

Corollary. Let $Q = I_1 \times \cdots \times I_n$, where I_j is a closed interval in $\mathbb{R}$ for each j. If $f : Q \to \mathbb{R}$ is continuous, then

$$\int_Q f = \int_{x_1 \in I_1} \cdots \int_{x_n \in I_n} f(x_1, \ldots x_n)$$

Partitions of Unity

Defn. Let $A \subset \mathbb{R}^n$, and let $\mathcal{O}$ be a collection of open sets that cover A. (i.e. $A \subset \bigcup_{V \in \mathcal{O}} V$). Consider a collection Φ of C^∞ functions defined on an open set $W \supset A$ ($\varphi \in \Phi$, $\varphi : W \to \mathbb{R}$) such that:

(a) For each $x \in A$, and each $\varphi \in \Phi$, we have

$$0 \leq \varphi(x) \leq 1.$$

(b) For each $x \in A$, $\exists$ an open set $V \ni x$ such that all but finitely many $\varphi \in \Phi$ are 0 on V.

(c) For each $x \in A$, we have $\sum_{\varphi \in \Phi} \varphi(x) = 1$.

(d) For each $\varphi \in \Phi$, $\exists$ an open set $U \in O$ such that $\varphi = 0$ outside some closed set contained in U.

Then:

(i) A collection Φ satisfying (a)–(c) is called a C^∞

Partition of unity for A.

(ii) If in addition, Φ also satisfies (d), it is said to be subordinate to the cover $\mathcal{O}$.

Theorem. Let $A \subset \mathbb{R}^n$, and let $\mathcal{O}$ be a collection of open sets that cover A.

Then there exists a C^∞ partition of unity Φ for A that is subordinate to the cover $\mathcal{O}$.

Proof.

Case 1. A is compact.

Then $\exists$ finitely many $\{U_i\}_{i=1}^{n}$, $U_i \in \mathcal{O}$ that cover A.

Claim. We can find compact sets $D_i \subset U_i$ such that $A = \bigcup_{i=1}^{n} D_i$

Proof (of claim). The sets D_i can be inductively constructed as follows:

Suppose that $D_1, \ldots, D_k$ have been chosen so that:

$$\underline{\left(\bigcup_{i=1}^{k} D_i^{\circ}\right) \cup \left(\bigcup_{i=k+1}^{n} U_i\right)}$$

covers A.

Let

$$C_{k+1} = A - \left(\left(\bigcup_{i=1}^{k} D_i^{\circ}\right) \cup \left(\bigcup_{i=k+1}^{n} U_i\right)\right)$$

Then $C_{k+1} \subset U_{k+1}$ is compact.

$\Rightarrow \exists$ a compact set D_{k+1} such that:

$\underline{C_{k+1} \subset D_{k+1}^{\circ}}$ and

$\underline{D_{k+1} \subset U_{k+1}}$ (why?)

claim

Let φ_i be a nonnegative C^∞- function which is positive on D_i and 0 outside of some closed set containing U_i. (Why?)

Since $\{D_1, \dots, D_n\}$ covers A, we have:

$$\sum_{i=1}^{n} \varphi_i(x) > 0, \quad \forall \; x$$

in some open set $U \supset A$.

On U, we define:

$$\varphi_i(x) = \frac{\psi_i(x)}{\sum_{i=1}^{n} \psi_i(x)}$$ ✓

If f is a C^∞-function which is 1 on A and 0 outside some closed set in U, then:

$\Phi = \{f\circ\varphi_1, \ldots, f\circ\varphi_n\}$ is the desired partition of unity

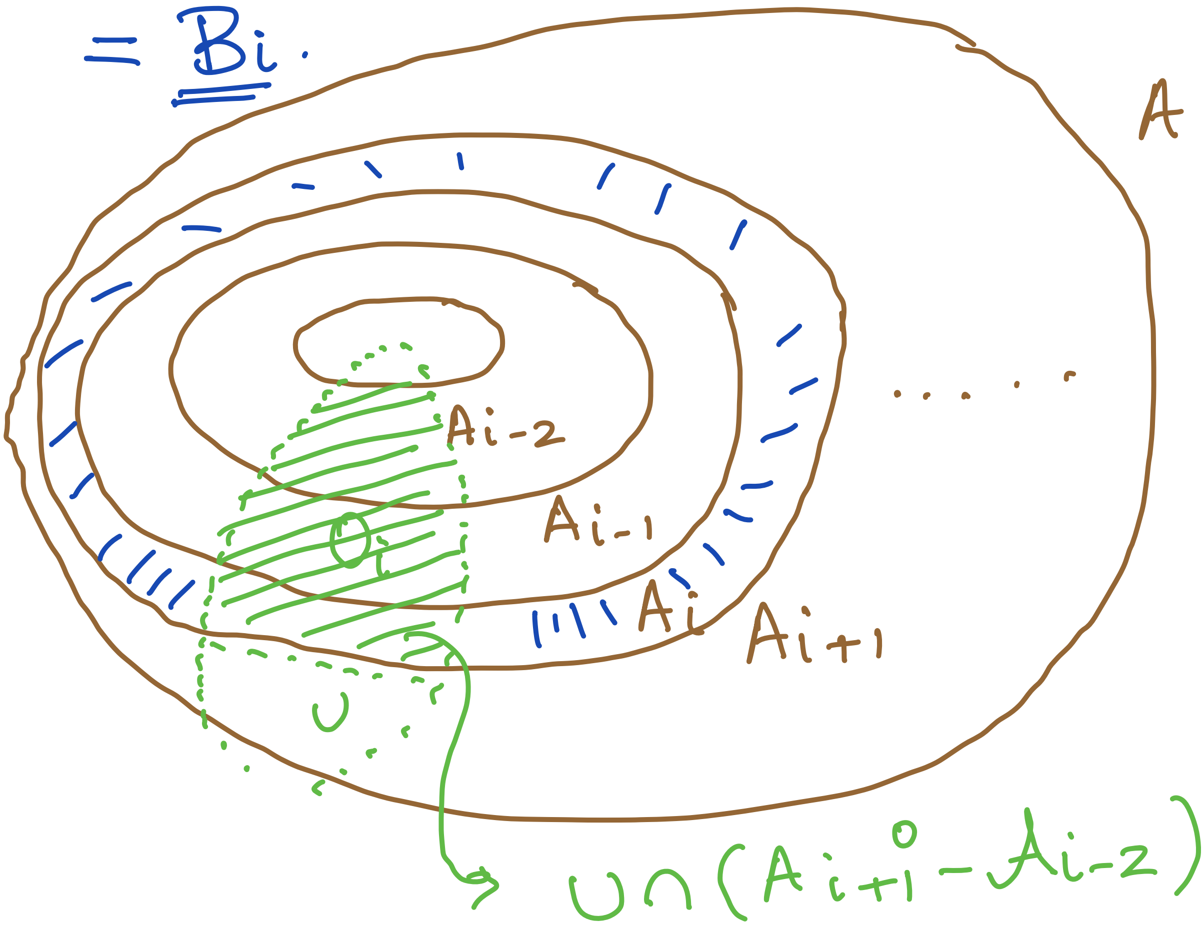
Case 2: Let $A = A_1 \cup A_2 \cup \ldots$ where each A_i is compact and $A_i \subset \mathring{A}_{i+1}$.

For each i, let:

$$\underline{\underline{O_i}} = \{ U \cap (A_{i+1}^{\circ} - A_{i-2}) : U \in O \}$$

Then $\{O_i\}$ covers $\underline{\underline{A_i - A_{i-1}^{\circ}}}$

$= \underline{\underline{B_i}}$.



There exists a partion of unity Φ_i for B_i, subordinate to O_i.
(by part (a))

For each $x \in A$, the sum

$$\sigma(x) = \sum_{\varphi \in \Phi_i, \forall i} \varphi(x) \checkmark$$

is well-defined as its finite in some open set $\ni x$.

($\because x \in A_i \Longrightarrow \varphi(x) = 0$, for all $\varphi \in \Phi_j$, for $j \geq i+2$)

Then

$$\left\{ \varphi'(x) = \frac{\varphi(x)}{\sigma(x)} : \varphi \in \Phi_i, \ \forall i \right\}$$

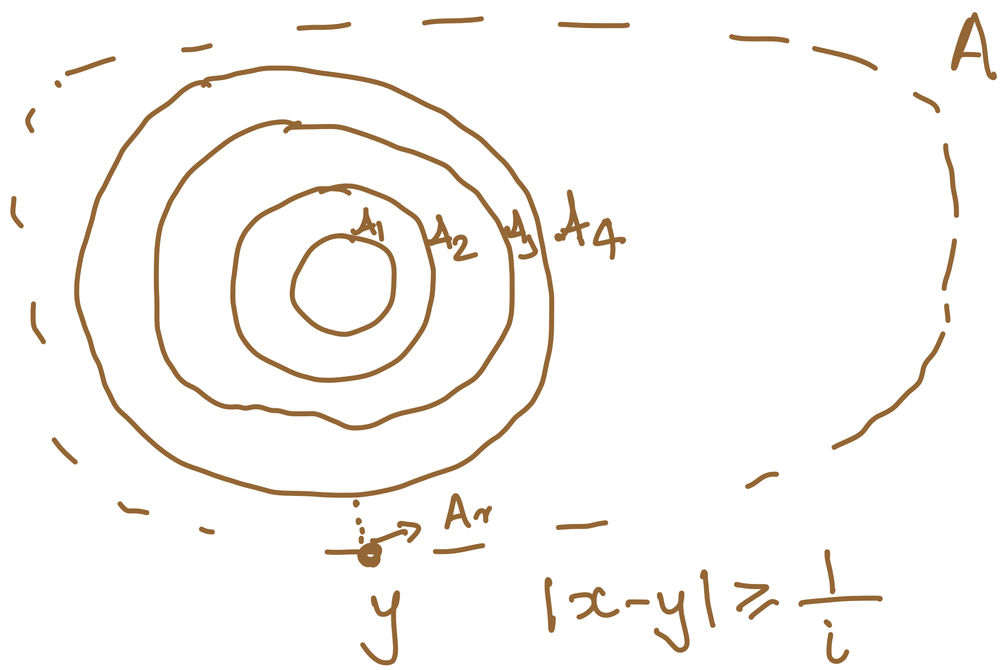
is the desired partition of unity.

<u>Case 3</u>. A is open.

Let

$$A_i = \left\{ x \in A : |x| \leq i \text{ and } \operatorname{dist}(x, \partial A) \geq \frac{1}{i} \right\}$$

Then $A = \bigcup_{i=1}^{\infty} A_i \quad (i \in \mathbb{N})$ (why?)



and $A_i \subset A_{i+1}$.

Since the A_i are compact.

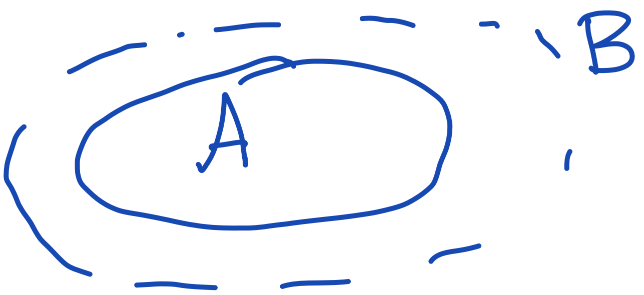
It follows now from Case (2).

<u>Case 4</u>. A is arbitrary

Let $B = \bigcup_{V \in O} V$. Then B

open set.

Then by Case(3), $\exists$ a POU for B, and hence for A. $\blacksquare$



<u>Remark</u>

Let $C \subset A$ be compact.
For each $x \in C$, $\exists$ an open set $V_x \ni x$ such that only finitely many $\varphi \in \Phi$ are nonzero on V_x. (from condition (b) of POU).

Since C is compact, finitely many such V_x cover C.

<u>Defn</u>. Let $\mathcal{O}$ be a (proper) open cover of $A \subset \mathbb{R}^n$. Let Φ be a PoU subordinate to $\mathcal{O}$. Let $f : A \longrightarrow \mathbb{R}$ be bounded in some open set around each point of A and $\{x \in A \mid f \text{ is discont.}\}$ has measure 0.

Then we say f is integrable on A if:

$$\sum_{\varphi \in \Phi} \int_A \underline{\underline{\varphi \cdot |f|}} \text{ converges.}$$

(By construction the function $\varphi \in \Phi$ can be arranged in sequence)

<u>Note</u>: This convergence

$$\Rightarrow \sum_{\varphi \in \Phi} \left| \int_A \varphi \cdot f \right| \text{ converges}$$

$$\left(\begin{array}{l} \varphi \cdot |f| = |\varphi \cdot f| \\ \left|\int g\right| \leq \int |g| \end{array} \right)$$

$$\Rightarrow \sum_{\varphi \in \Phi} \int_A \varphi \cdot f \text{ converges}$$

Defn So, we define

$$\sum_{\varphi \in \Phi} \int_A \varphi \cdot f := \int_A f \quad (*)$$

Theorem (a) If Ψ is another partition of unity subordinate to a (proper) cover $\mathcal{O}'$ of A, then

$$\sum_{\psi \in \Psi} \psi \cdot |f|$$ also converges,

and

$$\sum_{\varphi \in \Phi} \int_A \varphi \cdot f = \sum_{\psi \in \Psi} \int_A \psi \cdot f$$

((*) is well-defined)

(b) If A and f are bounded, then f is integrable on A.

(c) If A is bounded and ∂A has measure 0 (Jordan-measurable), then f is integrable on A.

Proof

(a) Note that $\varphi \cdot f = 0$ except on a compact $c(CA)$, and there exists only finitely many $\psi \in \Psi$ that are nonzero on C.

Therefore, we can write:

$$\sum_{\varphi \in \Phi} \int_A \varphi \cdot f = \sum_{\varphi \in \Phi} \int_A \sum_{\psi \in \Psi} \psi \cdot \varphi \cdot f$$

$$\int_A f = \sum_{\varphi \in \Phi} \sum_{\psi \in \Psi} \int_A (\psi \cdot \varphi \cdot f) \quad \text{(series)}$$

(**)

Apply (**) to $|f|$.

Then we have;

$$\sum_{\varphi\in\Phi}\sum_{\psi\in\Psi}\int_A \psi\cdot\varphi\cdot|f| \text{ converges}$$

$$\int_A \psi\cdot\varphi\cdot|f| = |\psi\cdot\varphi\cdot f|$$

$$\Rightarrow \underline{\sum_{\varphi\in\Phi}\sum_{\psi\in\Psi}\left|\int_A \psi\cdot\varphi\cdot f\right|} \quad (***)$$

converges

$$\left(\left|\int f\right| \le \int |f|\right)$$

obtained from $(**)$

Since $(***)$ absolutely converges $\Rightarrow$ Summations can be interchanged.

Upon interchanging the summation in $(**)$, we obtain:

$$\sum_{\varphi \in \Phi} \int_A \varphi \cdot f \text{ converges.}$$

Applying this to $|f|$, we get:

$$\sum_{\varphi \in \Phi} \int_A \varphi \cdot |f| \text{ converges}$$

$\blacksquare$ (a)

(b) If A is bounded, then A is contained in a closed rectangles $\underline{\underline{B}}$ and f is bdd $\Rightarrow$ $|f(x)| \leq M$, for $x \in A$.

Suppose that $F \subset \Phi$ is finite. Then

$$\sum_{\varphi \in F} \int_A \varphi \cdot |f| \leq \sum_{\varphi \in F} M \int_A \varphi$$

$$= M \int_A \sum_{\varphi \in F} \varphi$$

$$\leq M\, \nu(B)$$

$$\left(\sum_{\varphi \in F} \varphi \leq 1 \right)$$

(c) If A is Jordan-measurable and f is bounded, then f is integrable on A.

For $\varepsilon > 0$, $\exists$ compact $C \subset A$ such that

$$\int_{A \setminus C} 1 < \varepsilon \quad \text{(why? Exercise)}$$

Moreover, $\exists$ only finitely many $\varphi \in \Phi$ nonzero on C.

If $F \subset \Phi$ is any collection that includes these finitely many φs.

$$\left|\int_A f - \sum_{\varphi\in F}\int_A \varphi\cdot f\right| \checkmark$$

$$\leq \int_A \left| f - \sum_{\varphi\in F} \varphi\cdot f \right|$$

$$\leq M \int_A \left(1 - \sum_{\varphi\in F}\varphi\right)$$

(f is bounded by M)

$$= M \int_A \sum_{\varphi\in\Phi\setminus F}\varphi$$

$$\leq M \int_{A\setminus C} 1$$

$$\leq M\varepsilon \quad \blacksquare$$

Change of variables

If $g:[a,b] \longrightarrow \mathbb{R}$ is continuously diff. and $f: \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, then

$$\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g)\, g'$$

Moreore, when g is 1-1

$$\int_{g((a,b))} f = \int_{(a,b)} (f \circ g)\, |g'|$$

<u>Theorem</u>. Let $A \subset \mathbb{R}^n$ be an open set, and $g: A \to \mathbb{R}^n$ a 1-1 continuously differentiable function such that $\det\left(\overset{g'(x)}{\overset{\shortparallel}{Dg(x)}}\right) \neq 0$, for all $x \in A$. If $f: g(A) \to \mathbb{R}$ is integrable, then:

$$\int_{g(A)} f = \int_A (f\circ g)\left|\det\left(\overset{g'}{\overset{\shortparallel}{Dg}}\right)\right|$$

Proof.

Claim 1. Suppose that there exists a proper cover $\mathcal{O}$ for A such that for each $U \in \mathcal{O}$ and any integrable f (on A), we have:

$$\int_{g(U)} f = \int_U (f \circ g) |\det g'|$$

Then the Theorem holds for all A.

Proof (of Claim 1).

Note that $\{g(U) : U \in \mathcal{O}\}$ is an open cover of $g(A)$.

Let Φ be a PoU subordinate to this cover.

For $\varphi \in \Phi$, if $\varphi = 0$ outside $g(U)$, then $(\varphi \circ f) \circ g = 0$ outside U. (g is 1-1)

Therefore, this expression

$$\int_{g(U)} \varphi \cdot f = \int_U [(\varphi \cdot f) \circ g] \, |\det g'|$$

can be written as (i.e. is equivalent to)

$$\int_{g(A)} \varphi \cdot f = \int_A [(\varphi \cdot f) \circ g] \, |\det g'|$$

Hence,

$$\int_{g(A)} f = \sum_{\varphi \in \Phi} \int_{g(A)} \varphi \cdot f$$

$$= \sum_{\varphi \in \Phi} \int_A [(\varphi \cdot f) \circ g] |\det g'|$$

$$= \sum_{\varphi \in \Phi} \int_A (\varphi \circ g) \cdot (f \circ g) |\det g'|$$

$$= \int_A (f \circ g) |\det g'| \quad \blacksquare_1$$

<u>Claim 2</u>. It suffices to prove the Theorem for $f = 1$.

<u>Proof</u> (of Claim 2)[1]: If the Theorem holds true for $f = 1$, then it holds true for all

constant functions.

Let V be a rectangle in $g(A)$ and P a partition of V.

For each subrectangle S of P, let

$$f_S = m_S(f) \quad \left(\begin{array}{c}\text{inf of}\\ f \text{ over } S\end{array}\right)$$

$$L(f,P) = \sum_S m_S(f)\, v(S)$$

$$= \sum_S \int_{S^\circ} f_S$$

$$= \sum_S \int_{g^{-1}(S^\circ)} (f_S \circ g)\,|\det g'|$$

(Theorem holds for const. fn f_S)

$$\leq \sum_{S} \int_{g^{-1}(S^\circ)} (f\circ g)|\det g'|$$

$$\leq \int_{g^{-1}(V)} (f\circ g)|\det g'|$$

By defn, $\int_V f$ is the LUB for all $L(f,P)$. So we have

$$\int_V f \leq \int_{g^{-1}(V)} (f\circ g)|\det g'| \qquad L\text{(1)}$$

In a similar manner, we can ~~take~~ take $f_S = M_S(f)$ and repating the arguments above, to obtain:

$$\int_V f \geq \int_{g^{-1}(V)} (f \circ g)\,|\det g'| \quad \text{L } 2$$

$\Rightarrow$ From 1 & 2, we have:

$$\int_V f = \int_{g^{-1}(V)} (f \circ g)\,|\det g'|$$

for V in some proper cover of $g(A)$.

The result now follows from Claim 1 ~~&~~ 2

Claim 3. If the theorem holds for $g: A \to \mathbb{R}^n$ and $h: B \to \mathbb{R}^n$, where $g(A) \subset B$, then it holds for

$h \circ g : A \longrightarrow \mathbb{R}^n$

Proof (Claim 3). Exercise

Claim 4. The theorem holds when g is a linear transformation.

Proof (Claim 4). From Claims 1 and 2, it suffices to show for any open rectangle U that

$$\int_{g(U)} 1 = \int_U |\det g'|$$

(Exercise) $\equiv$ claim 4

Proof (contd.) We prove the main Theorem by induction on n.

The theorem clearly holds for $n=1$. (due to Claim 1 & 2 and the 1-dim^{l} case).

Assume that the theorem holds for $(n-1)$.

We show that it holds for n.

For each $a \in A$, we need to find an open set $U \ni a$ $(U \subset A)$ for which the theorem holds.

Moreover, we may assume WLOG $g'(a) = I$. (?)

Define $h : A \longrightarrow \mathbb{R}^n$ by

$$x = (x_1, \cdots x_n) \xmapsto{h} (g_1(x), \ldots, g_{n-1}(x), x_n)$$

Then $h'(a) = I$

Hence, in some nbhd $a \in U' \subset A$ h is 1-1 and $\det(h'(x)) \neq 0$.

Define

$$k : h(U') \longrightarrow \mathbb{R}^n$$

By

$$k(x) = (x_1, \dots, x_{n-1}, g_n(h^{-1}(x)))$$

Then $g = k \circ h$.

Since

$$(g_n \circ h)^{-1}(h(a))$$

$$= (g_n)'(a)(h'(a))^{-1}$$

$= (gh)'(a) \quad [h'(a) = I]$

Thus, in some nbhd $h(a) \in V \subset h(U')$, k is 1-1 and $\det(Dk(x)) \neq 0$.

$\underset{\text{''}}{k'(x)}$

Putting $U = k^{-1}(V)$, we have $g = k \circ h$, where $h : U \longrightarrow \mathbb{R}^n$ and $k : V \longrightarrow \mathbb{R}^n$ with $h(U) \subset V$.

We establish the assertion for h (as the proof for k is similar)

Let $W \subset U$ be a rectangle of the form $D \times [a_n, b_n]$, where $D \subset \mathbb{R}^{n-1}$

$$\int_{h(W)} 1 = \int_{[a_n, b_n]} \left(\int_{h(D \times \{x_n\})} 1 \, dx_1 \cdots dx_{n-1} \right) dx_n$$

Let

$h_{x_n} : D (\subset \mathbb{R}^{n-1}) \longrightarrow \mathbb{R}^{n-1}$ be defined by

$h_{x_n}(x_1, \ldots, x_{n-1})$

$= (g_1(x_1, \ldots x_n), \ldots, g_{n-1}(x_1, \ldots, x_n))$

Then each h_{x_n} is 1-1

$\Rightarrow \det(Dh_{x_n})(x_1, \ldots, x_{n-1})$

$$= \det(Dh(x_1, \dots, x_n)) \neq 0$$

$$(g = k \circ h)$$

Hence,

$$\int_{h(D \times \{x_n\})} 1 = \int_{h_{x^n}(D)} 1$$

Applying the theorem in the $(n-1)$-case

$$\int_{h(W)} 1 = \int_{[a_n, b_n]} \left(\int_{h_{x^n}(D)} 1 \, dx_1 \dots dx_{n-1} \right) dx_n$$

$$= \int_{[a_n, b_n]} \left(\int_D |\det Dh_{x^n}(x_1, \dots x_{n-1})| \, dx_1 \dots dx_{n-1} \right) dx_n$$

$$= \int_{[a_n, b_n]} \left(\int_D |\det Dh(x_1, \dots, x_n)| \, dx_1 \dots dx_{n-1} \right) dx_n$$

$$= \int_W |\det Dh(x)| .$$

Theorem (Sard's Theorem). Let $g : A \longrightarrow \mathbb{R}^m$ be continuously differentiable, where $A \subset \mathbb{R}^n$ is open, and let

$$B = \{ x \in A : \det(Dg(x)) = 0 \}.$$

Then $g(B)$ has measure zero.

Integration on chains

Multilinear algebra

Defn. Let V be a vector space over $\mathbb{R}$, and let $V^k = V \times \cdots \times V$ be the k-fold product. A function $T : V^k \to \mathbb{R}$ is said to be multilinear if for each i with $1 \le i \le k$, we have:

(a) $T(v_1, \ldots, v_i + v_i', \ldots v_k)$
$$= T(v_1, \ldots v_i, \ldots v_k) + T(v_1, \ldots, v_i', \ldots v_k)$$

(b) $T(v_1, \dots, av_i, \dots v_k)$

$$= aT(v_1, \dots v_i, \dots v_k)$$

<u>Defn</u>. A multilinear function $T: V^k \to \mathbb{R}$ is called a <u>k-tensor</u> on V.

<u>Remark</u>. The set of all k-tensors $\mathcal{T}^k(V)$ on V is a vector space over $\mathbb{R}$

<u>Defn</u>. For $S \in \mathcal{T}^k(V)$ and $T \in \mathcal{T}^l(V)$, we define the tensor product $S \otimes T \in \mathcal{T}^{k+l}(V)$

by :

$$S \otimes T(v_1, \dots, v_k, v_{k+1}, \dots, v_\ell)$$
$$= S(v_1, \dots, v_k) \cdot T(v_{k+1}, \dots, v_\ell)$$

<u>Remark</u>. Note that

$$S \otimes T \neq T \otimes S$$

<u>Lemma</u>. Tensor product $\otimes$ satisfies the following properties.

(a) $(S_1 + S_2) \otimes T = S_1 \otimes T + S_2 \otimes T$

(b) $S \otimes (T_1 + T_2) = S \otimes T_1 + S \otimes T_2$

(c) $(aS) \otimes T = S \otimes aT = a(S \otimes T)$

(d) $(S \otimes T) \otimes U = (S \otimes T) \otimes U$

Remark

(i) The tensor products in (d) are usually denoted by $S \otimes T \otimes U$; higher products $T_1 \otimes \cdots \otimes T_r$ are defined similarly.

(ii) $\mathcal{T}^1(V) = V^*$ (dual space)

Theorem. Let $v_1, \dots, v_n$ be a basis for V, and let $\varphi_1, \dots, \varphi_n$ be basis for V^* so that $\varphi_i(v_j) = \delta_{ij}$. Then the set of all k-fold tensor products

$$\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} \quad 1 \le i_1, \dots, i_k \le n$$

is a basis for $\mathcal{T}^k(V)$.

Consequently, $\dim(\mathcal{T}^k(V)) = n^k$.

Proof

Observe that

$$(\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})(v_{j_1}, \ldots, v_{j_k})$$

$$= \delta_{i_1, j_1} \cdots \delta_{i_k, j_k}$$

$$= \begin{cases} 1, & \text{if } j_r = i_r, \text{ for } 1 \le r \le k \\ 0, & \text{otherwise.} \end{cases}$$

If $w_1, \ldots, w_k$ are k vectors with $w_i = \sum_{j=1}^{n} a_{ij} v_j$ and $T \in \mathcal{T}^k(V)$, then:

$$T(w_1, \ldots, w_k) = \sum_{j_1, \ldots, j_k = 1}^{n} a_{1,j_1} \cdots a_{k,j_k} \, T(v_{j_1}, \ldots v_{j_k})$$

$$= \sum_{i_1,\dots i_k=1}^{n} T(v_{i_1},\dots v_{i_k}) \cdot (\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})(w_1,\dots,w_k)$$

$$\Rightarrow T = \sum_{i_1,\dots i_k=1}^{n} T(v_{i_1},\dots v_{i_k}) \cdot (\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})$$

$\Rightarrow \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}$ span $\mathcal{T}^k(V)$.

Now suppose that

$$\sum_{i_1,\dots i_k=1}^{n} a_{i_1,\dots,i_k} \cdot \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} = 0$$

Apply both sides to $(v_{j_1},\dots,v_{j_k})$, we have:

$$a_{j_1,\dots j_k} = 0 \quad \blacksquare$$

Remark. If $f: V \longrightarrow W$ is a linear transformation, then

$f^*: \mathcal{T}^k(W) \longrightarrow \mathcal{T}^k(V)$

defined by:

$$f^* T(v_1, \ldots, v_k) = T(f(v_1), \ldots, f(v_k))$$

for $T \in \mathcal{T}^k(W)$ and $v_1, \ldots, v_k \in V$, is also a linear transformation.

Check: $f^*(S \otimes T) = f^*S \otimes f^*T$.

Examples.

(a) An inner product T on V ($T: V \times V \longrightarrow \mathbb{R}$) is a 2-tensor (i.e. $T \in \mathcal{T}^2(V)$)

that is:

(i) <u>Symmetric</u>: $T(v,w)=T(w,v)$ for all $v,w \in V$, and

(ii) <u>Positive definite</u>: $T(v,v) \geq 0$, for all $v \in V$.

<u>Theorem</u>. If T is an inner product on V, there exists a basis $v_1, \dots, v_n$ for V such that $T(v_i, v_j) = \delta_{ij}$. (i.e. an orthonormal basis). Consequently, $\exists$ an isomorphism $f: \mathbb{R}^n \longrightarrow V$ such that $T(f(x,y)) = \langle x,y \rangle$, for $x,y \in \mathbb{R}^n$.

where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^n$. In other words $f^*T = \langle , \rangle$

<u>Proof</u>. Let $w_1, \dots, w_n$ is a basis for V. Then define.

$$w_1' = w_1$$

$$w_2' = w_2 - \frac{T(w_1', w_2)}{T(w_1', w_1')} \cdot w_1'$$

$$w_3' = w_3 - \frac{T(w_1', w_3)}{T(w_1', w_1')} \cdot w_1' - \frac{T(w_2', w_3)}{T(w_2', w_2')} \cdot w_2'$$

$$\vdots$$

Then $T(w_i', w_j') = 0$ if $i \neq j$ and $w_i' \neq 0$ so that $T(w_i', w_i') > 0$.

Defining $v_i = \frac{w_i'}{|w_i|}$, the map $e_i \xmapsto{f} v_i$ is an isomorphism ∎

<u>Defn</u>. A k-tensor $\omega \in \mathcal{J}^k(V)$ is called alternating if

$$\omega(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\omega(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k),$$

for all $v_1, \ldots, v_k \in V$.

The set of all alternating tensors is a subspace of $\mathcal{J}^k(V)$ denoted by $\Lambda^k(V)$.

We define

$$\mathrm{Alt}(T)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn} \sigma \cdot T(v_{\sigma(1)}, \ldots, v_{\sigma(k)}),$$

where S_k is permutation group of $\{1, 2, \ldots, k\}$

Theorem.

(1) If $T \in \mathcal{T}^k(V)$, then $\operatorname{Alt}(T) \in \Lambda^k(V)$.

(2) If $\omega \in \Lambda^k(V)$, then $\operatorname{Alt}(\omega) = \omega$.

(3) If $T \in \mathcal{T}^k(V)$, then $\operatorname{Alt}(\operatorname{Alt}(T)) = \operatorname{Alt}(T)$.

Proof

(1) Consider the transposition $(i\,j) \in S_k$, and let $\sigma' = \sigma \circ (i\,j)$ for each $\sigma \in S_k$.

Then

$$\operatorname{Alt}(T)(v_1, \dots, v_j, \dots, v_i, \dots, v_k)$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) T(v_{\sigma(1)}, \dots, v_{\sigma(j)}, \dots, v_{\sigma(i)}, \dots, v_{\sigma(k)})$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) T(v_{\sigma'(1)}, \dots, v_{\sigma'(i)}, \dots, v_{\sigma'(j)}, \dots, v_{\sigma'(k)})$$

$$= \frac{1}{k!} \sum_{\sigma' \in S_k} -\operatorname{sgn}(\sigma) T(v_{\sigma'(1)}, \dots, v_{\sigma'(k)})$$

$$= -\operatorname{Alt}(T)(v_1, \dots, v_k)$$

(2) If $\omega \in \Lambda^k(V)$ and $\sigma = (i,j)$, then $\omega(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma) \cdot \omega(v_1, \dots, v_k)$. (*)

Since every $\sigma \in S_k$ is a product of transpositions, (*) holds for all $\sigma \in S_k$.

Therefore,

$\mathrm{Alt}(\omega)(v_1,\dots,v_k)$

$$= \frac{1}{k!} \sum_{\sigma\in S_k} \mathrm{sgn}(\sigma)\cdot \omega(v_{\sigma(1)},\dots,v_{\sigma(k)})$$

$$= \frac{1}{k!} \sum_{\sigma\in S_k} \mathrm{sgn}(\sigma)\cdot \mathrm{sgn}(\sigma)\cdot \omega(v_1,\dots,v_k)$$

$$= \omega(v_1,\dots,v_k).$$

(3) Follows from 1 & 2.

<u>Note</u>. $\omega\in\Lambda^k(V)$ and $\eta\in\Lambda^\ell(V)$

$\not\Rightarrow \omega\otimes\eta\in\Lambda^{k+\ell}(V)$.

<u>Defn</u>. For $\omega\in\Lambda^k(V)$ and $\eta\in\Lambda^\ell(V)$, we define the <u>wedge product</u> by:

$$\omega\wedge\eta := \frac{(k+\ell)!}{k!\,\ell!}\,\mathrm{Alt}(\omega\otimes\eta)$$

Lemma. Wedge product satisfies the following properties:

(a) $(\omega_1+\omega_2)\wedge\eta = \omega_1\wedge\eta + \omega_2\wedge\eta$

(b) $\omega\wedge(\eta_1+\eta_2) = \omega\wedge\eta_1 + \omega\wedge\eta_2$

(c) $a\omega\wedge\eta = \omega\wedge a\eta = a(\omega\wedge\eta)$

(d) $\omega\wedge\eta = (-1)^{kl}\,\eta\wedge\omega$

(e) $f^*(\omega\wedge\eta) = f^*(\omega)\wedge f^*(\eta)$

Theorem

(1) If $S\in\mathcal{J}^k(V)$ and $T\in\mathcal{J}^l(V)$ and $\mathrm{Alt}(S)=0$, then

$$\mathrm{Alt}(S\otimes T) = \mathrm{Alt}(T\otimes S) = 0$$

(2) $$\mathrm{Alt}(\mathrm{Alt}(\omega\otimes\eta)\otimes\theta) = \mathrm{Alt}(\omega\otimes\eta\otimes\theta)$$

$$= \mathrm{Alt}(\omega \otimes \mathrm{Alt}(\eta \otimes \theta))$$

(3) If $\omega \in \Lambda^k(V)$, $\eta \in \Lambda^l(V)$, and $\theta \in \Lambda^m(V)$, then

$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta)$$
$$= \frac{(k+l+m)!}{k!\,l!\,m!} \mathrm{Alt}(\omega \otimes \eta \otimes \theta)$$

Proof.

$$(k+l)!\, \mathrm{Alt}(S \otimes T)(v_1, \ldots, v_{k+l})$$
$$= \sum_{\sigma \in S_{k+l}} \operatorname{sgn}\sigma \cdot S(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \cdot T(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)})$$

Let $G \subset S_{k+l}$ consist of all σ that fix $k+1, \ldots, k+l$.

Then

$$\sum_{\sigma \in G} \operatorname{sgn} \sigma \cdot S(v_{\sigma(1)}, \dots v_{\sigma(k)}) \cdot T(v_{\sigma(k+1)} \dots v_{\sigma(k+l)})$$

$$= \left[\sum_{\sigma' \in S_k} \operatorname{sgn} \sigma' S(v_{\sigma'(1)}, \dots, v_{\sigma'(k)})\right] \cdot T(v_{k+1}, \dots, v_{k+l})$$

Now let $\sigma \in S_{k+l} \setminus G$, let $G \cdot \sigma_0 = \{\sigma \cdot \sigma_0 \mid \sigma \in G\}$, and

$v_{\sigma_0(1)}, \dots v_{\sigma_0(k+l)} = w_1, \dots, w_{k+l}$.

Then

$$\sum_{\sigma \in G \cdot \sigma_0} \operatorname{sgn} \sigma . S(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \cdot T(v_{\sigma(k+1)}, \dots v_{\sigma(k+l)})$$

$$= \left[\operatorname{sgn} \sigma_0 \cdot \sum_{\sigma' \in G} \operatorname{sgn} \sigma' S(w_{\sigma'(1)} \dots w_{\sigma'(k)}) \cdot T(w_{k+1}, \dots, w_{k+l})\right.$$

$$= 0.$$

(Note that $G \cap G \cdot \sigma_0 = \emptyset$).

(2) We have

$$\mathrm{Alt}(\mathrm{Alt}(\eta \otimes \theta) - \eta \otimes \theta)$$
$$= \mathrm{Alt}(\eta \otimes \theta) - \mathrm{Alt}(\eta \otimes \theta)$$

$\Rightarrow$ By (1), we have

$$0 = \mathrm{Alt}(\omega \otimes [\mathrm{Alt}(\eta \otimes \theta) - \eta \otimes \theta])$$
$$= \mathrm{Alt}(\omega \otimes \mathrm{Alt}(\eta \otimes \theta))$$
$$- \mathrm{Alt}(\omega \otimes \eta \otimes \theta)$$

(3) $(\omega \wedge \eta) \wedge \theta$

$$= \frac{(k+l+m)!}{(k+l)!\, m!} \mathrm{Alt}((\omega \wedge \eta) \otimes \theta)$$

$$= \frac{(k+l+m)!}{(k+l)!\, m!} \frac{(k+l)!}{k!\, l!} \mathrm{Alt}(\omega \otimes \eta \otimes \theta)$$

We denote both $\omega\wedge(\eta\wedge\theta)$ and $(\omega\wedge\eta)\wedge\theta$ by $\omega\wedge\eta\wedge\theta$.

Higher-order products are denoted by $\omega_1\wedge\omega_2\wedge\cdots\wedge\omega_r$.

<u>Theorem</u>. The set of all

$\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_k}$ $\quad 1\le i_1<i_2<\cdots<i_k\le n$

is a basis for $\Lambda^k(V)$.

Consequently,

$$\dim \Lambda^k(V) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

<u>Theorem</u>. Let $v_1,\dots,v_n$ be a basis for V, and let $\omega\in\Lambda^n(V)$. If $w_i = \sum_{j=1}^{n} a_{ij}v_j$ for $1\le i\le n$, then:

$$\omega(w_1, \dots w_n) = \det(a_{ij})\,\omega(v_1, \dots, v_n).$$

<u>Proof</u>. Define $\eta \in \mathcal{J}^n(\mathbb{R}^n)$ by

$$\eta((a_{11}, \dots, a_{1n}), \dots, (a_{n1}, \dots, a_{nn}))$$
$$= \omega\left(\sum a_{1j} v_j, \dots, \sum a_{nj} v_j\right)$$

Then $\eta \in \Lambda^n(\mathbb{R}^n)$ and

$$\eta = \eta(e_1, \dots, e_n) \cdot \det(a_{ij})$$
$$= \omega(v_1, \dots v_n) \cdot \det(a_{ij}) \blacksquare$$

<u>Remark</u>. By theorem, a nonzero $\omega \in \Lambda^n(V)$ splits bases of V into two groups:

(a) Those with $\omega(v_1, \dots, v_n) < 0$

(b) Those with $\omega(v_1, \dots, v_n) > 0$.

Two bases $v_1, \dots, v_n$ and $w_1, \dots, w_n$ are in the same group if given $w_i = \sum a_{ij} v_j$, then $\det(a_{ij}) > 0$.

Defn. Either of these two groups is called an orientation for V.

In $\mathbb{R}^n$, the usual orientation is $[e_1, \dots, e_n]$.

Remark. (a) Note that $\dim \Lambda^n(\mathbb{R}^n) = 1$. In fact, det is often seen as the unique $\omega \in \Lambda^n(\mathbb{R}^n)$ such that $\omega(e_1, \dots, e_n) = 1$

Why? Suppose that T is an inner product and $v_1, \dots v_n; w_1, \dots w_n$

are two bases which are orthonormal with respect to T with $w_i = \sum_{j=1}^{n} a_{ij} v_j$.

Then
$$\delta_{ij} = T(w_i, w_j) = \sum_{k,l=1}^{n} a_{ik} a_{jl} T(v_k, v_l)$$
$$= \sum_{k=1}^{n} a_{ik} a_{jk}.$$

$$\Rightarrow \quad A \cdot A^T = I \Rightarrow \det(A) = \pm 1.$$

By theorem, if $\omega \in \Lambda^n(V)$ satisfies $\omega(v_1, \ldots, v_n) \pm 1$, then $\omega(w_1, \ldots, w_n) = \pm 1$.

If an orientation μ for V has been given, then $\exists!\ \omega \in \Lambda^n(V)$ such that $\omega(v_1, \dots, v_n) = 1$, whenever $v_1, \dots, v_n$ is an orthonormal basis such that $[v_1, \dots, v_n] = \mu$.

<u>Defn.</u>

This unique ω is called the <u>volume element</u> of V, determined by T and μ.

<u>Example</u> det is the volume element of $\mathbb{R}^n$ with $\langle\ \rangle$ and $[e_1, \dots, e_n]$.

In fact, $|\det(v_1, \dots, v_n)|$ = volume of parallelopiped spanned by $v_1, \dots, v_n$.

Defn. Let $v_1, \dots, v_{n-1} \in \mathbb{R}^n$ and φ is defined by

$$\varphi(w) = \det \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ w \end{pmatrix}$$

Then $\varphi \in \Lambda^1(\mathbb{R}^n)$ and $\exists!\, z \in \mathbb{R}^n$ such that

$$\langle w, z \rangle = \varphi(w) = \det \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ w \end{pmatrix}$$

This z is denoted by $v_1 \times \cdots \times v_{n-1}$ and is called the cross-product of $v_1, \dots, v_{n-1}$.

Lemma (a) $v_{\sigma(1)} \times \cdots \times v_{\sigma(n-1)} = \operatorname{sgn}\sigma \cdot (v_1 \times \cdots \times v_{n-1})$

(b) $v_1 \times \cdots \times a v_i \times \cdots v_{n-1} = a \cdot (v_1 \times \cdots \times v_n)$

(c) $v_1 \times \cdots \times (v_i + v_i') \times \cdots \times v_{n-1}$

$$= v_1 \times \cdots v_i \times \cdots v_{n-1} + v_1 \times \cdots \times v_i' \times \cdots \times v_{n-1}.$$

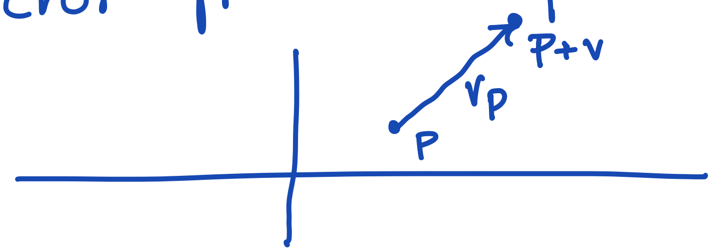
Vector fields and Differential Forms

Defn. For $p \in \mathbb{R}^n$, the tangent space of $\mathbb{R}^n$ at p is defined by $\mathbb{R}^n_p = \{(p,v) : v \in \mathbb{R}^n\}$.

Remark.
$\mathbb{R}^n_p$ is a vector space with respect to:

$$(p,v) + (p,w) = (p, v+w)$$
$$a \cdot (p,v) = (p, av)$$

Given p and $v \in \mathbb{R}^n_p$, we write $v_p = (v,p)$ and visualize it as a vector from the point p to $p+v$



The standard inner product $\langle . \rangle$ on $\mathbb{R}^n$ induces an inner product $\langle , \rangle_p$ on $\mathbb{R}^n_p$ define by $\langle u_p, v_p \rangle_p = \langle u, v \rangle$

Defn. A <u>vector field</u> is a function $F : \mathbb{R}^n \longrightarrow \bigcup_{x \in \mathbb{R}^n} \mathbb{R}^n_x$ such that $F(x) \in \mathbb{R}^n_p$, for each $p \in \mathbb{R}^n$.

Remark.
For each $p \in \mathbb{R}^n$, $\exists$ $F_1(p), \ldots, F_n(p)$ such that

$$F(p) = \sum_{i=1}^{n} F_i(p)(e_i)_p,$$

where the F_i are the component functions.

Defn A vector field F is continuous (resp. diff) if each F_i is continuous (resp. diff).

Defn. If F, G are vector fields, and f is a function, we define:

(a) $(F+G)(p) = F(p)+G(p)$

(b) $\langle F, G\rangle(p) = \langle F(p), G(p)\rangle$

(c) $(f \cdot F)(p) = f(p)\, F(p)$

Defn. If F_i, $1 \leq i \leq n$, are vector fields, we define:

$$(F_1 \times \cdots \times F_{n-1})(p) = F_1(p) \times \cdots \times F_{n-1}(p)$$

Defn We define the <u>divergence</u> of a vector field F by

$$\operatorname{div}(F) = \sum_{i=1}^{n} D_i F_i$$

In symbols, if $\nabla = \sum_{i=1}^{n} D_i \cdot e_i$, then $\operatorname{div}(F) = \langle \nabla, F \rangle$

Defn. Under this symbolism, we define the <u>curl</u> of F as the vector field

$$(\nabla \times F)(p) = \begin{vmatrix} (e_1)_p & (e_2)_p & (e_3)_p \\ D_1 & D_2 & D_3 \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Defn. A function

$$\omega : \mathbb{R}^n \longrightarrow \bigcup_{x \in \mathbb{R}^n} \Lambda^k(\mathbb{R}^n_x)$$

such that $\omega(p) \in \Lambda^k(\mathbb{R}^n_p)$, for each $p \in \mathbb{R}^n$ is called a <u>differentiable k-form</u> on $\mathbb{R}^n$

If $\varphi_1(p), \dots, \varphi_n(p)$ is a dual basis to $(e_1)_p, \dots, (e_n)_p$, then

$$\omega(p) = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k}(p) \cdot [\varphi_{i_1}(p) \wedge \dots \wedge \varphi_{i_k}(p)]$$

Remark (a) The operations $\omega + \eta$, $f \cdot \omega$, $\omega \wedge \eta$ are well-defined.

(b) A function f is considered to be a 0-form.

If $f:\mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable, then $Df(p) \in \Lambda^1(\mathbb{R}^n)$. So we define df by:

$$df(p)(v_p) = Df(p)(v)$$

For any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, let

$$x \xmapsto{\pi_i} x_i$$

Then

$$dx_i(p)(v_p) = d\pi_i(p)v_p = D\pi_i(p)(v)$$
$$= v_i$$

(Here we view x_i as π_i)

So, $dx_1(p), \dots, dx_n(p)$ is a dual basis to $(e_1)_p, \dots, (e_n)_p$.

Thus, every k-form can be written as

$$\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

Theorem. If $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable, then

$$df = D_1 f \cdot dx_n + \dots + D_n f \cdot dx_n$$

i.e. in classical notation,

$$df = \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n$$

$$\left(dx_i(p) = d\,\pi_i(p) \right)$$

Proof.

$$\begin{aligned} df_p(v_p) &= Df(p)(v) \\ &= \sum_{i=1}^{n} v_i D_i f(p) \\ &= \sum_{i=1}^{n} dx_i(p) v_p \cdot D_i f(p) \quad \blacksquare \end{aligned}$$

Consider $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $Df(p): \mathbb{R}^m \longrightarrow \mathbb{R}^n$. Then $f_* : \mathbb{R}^n_p \longrightarrow \mathbb{R}^m_{f(p)}$ is defined by

$$f_*(v_p) = (Df(p)(v))_{f(p)}.$$

This linear map induces a linear map

$$f^* : \Lambda^k(\mathbb{R}^m_{f(p)}) \longrightarrow \Lambda^k(\mathbb{R}^n_p)$$

If ω is a k-form on $\mathbb{R}^m$, we define a k-form $f^*\omega$ on $\mathbb{R}^n$ by:

$$(f^*\omega)(p) = f^*(\omega(f(p)))$$

i.e. if $v_1, \dots, v_k \in \mathbb{R}^n_p$, then

$$(f^*\omega)(p)(v_1, \dots, v_k)$$
$$= \omega(f(p))(f_*(v_1), \dots, f_*(v_k))$$

Theorem. If $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable, then:

(a) $f^*(dx_i) = \sum_{j=1}^{n} D_j f_i \cdot dx_j$

$$= \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} dx_j$$

(b) $f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2)$

(c) $f^*(g \cdot \omega) = (g \circ f) \cdot f^*\omega$

(d) $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$

Proof

(a)

$$f^*(dx_i)(p)(v_p) = dx_i(f(p))(f_* v_p)$$

$$= dx_i(f(p))\left(\sum_{j=1}^{n} v_j D_j f_1(p), \ldots, \sum_{j=1}^{n} v_j D_j f_m(p)\right)_{f(p)}$$

$$= \sum_{j=1}^{n} v_j D_j f_i(p)$$

$$= \sum_{j=1}^{n} D_j f_i(p) \cdot dx_j(p)(v_p) \quad \blacksquare$$

Theorem. If $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is differentiable, then

$$f^*(h\, dx_1 \wedge \cdots \wedge dx_n) = (h \circ f)(\det f')\, dx_1 \wedge \cdots \wedge dx_n$$

Proof. Since

$$f^*(h\, dx_1 \wedge \cdots \wedge dx_n) = (h \circ f)\, f^*(dx_1 \wedge \cdots \wedge dx_n),$$

it suffices to show that

$$f^*(dx_1 \wedge \cdots \wedge dx_n) = \det(Df)\, dx_1 \wedge \cdots \wedge dx_n$$

Let $p \in \mathbb{R}^n$ and let $A = (a_{ij}) = Df(p)$

Then

$$f^*(dx_1 \wedge \cdots \wedge dx_n)(e_1, \ldots, e_n) = dx_1 \wedge \cdots \wedge dx_n(f_* e_1, \ldots, f_* e_n)$$

$$= dx_1 \wedge \cdots \wedge dx_n \left(\sum_{i=1}^{n} a_{i1} e_i, \ldots, \sum_{i=1}^{n} a_{in} e_i \right)$$

$$= \det(a_{ij}) \cdot dx_1 \wedge \cdots \wedge dx_n (e_1, \ldots, e_n)$$

$\blacksquare$

<u>Defn</u> Given the k-form

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

we define a $(k+1)$-form $d\omega$, the differential of ω, by

$$d\omega = \sum_{i_1 < \cdots < i_k} d\omega_{i_1 \ldots i_k} \wedge dx_{i_1} \wedge \cdots dx_{i_k}$$

$$= \sum_{i_1 < \cdots < i_k} \sum_{\alpha=1} D_\alpha(\omega_{i_1, \ldots, i_k}) \cdot dx_\alpha \wedge dx_{i_1} \wedge \cdots dx_{i_k}$$

<u>Theorem</u>.

(i) $d(\omega + \eta) = d\omega + d\eta$

(ii) If ω is a k-form and η is a 1-form, then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(iii) $d(d\omega)=0$ (i.e. $d^2=0$)

(iv) If ω is a k-form on $\mathbb{R}^m$ and $f:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ is diff, then $f^*(d\omega) = d(f^*\omega)$.

<u>Defn</u>. A form ω is <u>closed</u> if $d\omega=0$ and <u>exact</u> if $\omega = d\eta$, for some η.

<u>Remark</u> (i) By theorem, every exact form is closed.

Conversely, if $\omega = P\,dx + Q\,dy$ is a 1-form in $\mathbb{R}^2$, then

$$d\omega = (D_1P\,dx + D_2P\,dy)\wedge dx + (D_1Q\,dx + D_2Q\,dy)\wedge dy$$
$$= (D_1Q - D_2P)\,dx\wedge dy$$

So, if $d\omega = 0$, then

$$D_1 Q = D_2 P.$$

$\exists$ a function f such that

$\omega = df = D_1 f\, dx + D_2 f\, dy$. (HW)

(ii) However, if ω is defined only on a subset of $\mathbb{R}^2$

For example, consider

$$\omega = \frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy$$

on $\mathbb{R}^2 - \{0\}$

Then $\omega = d\theta$, where

$$\theta(x,y) = \begin{cases} \tan^{-1}(y/x) & x, y > 0 \\ \pi + \tan^{-1}(y/x) & x < 0 \\ 2\pi + \tan^{-1}(y/x) & x > 0, y < 0 \\ \pi/2 & x = 0, y > 0 \\ 3\pi/2 & x = 0, y < 0 \end{cases}$$

which is not continuous on $\mathbb{R}^2 - \{0\}$.

If $\omega = df$, for some $f : \mathbb{R}^2 - 0 \to \mathbb{R}$, then $D_1 f = D_1\theta$ and $D_2 f = D_2\theta$

$\Rightarrow f = \theta + c \Rightarrow f$ cannot exist.

(iii) Suppose that $\omega = \sum_{i=1}^{n} \omega_i dx_i$ is a 1-form on $\mathbb{R}^n$ and

$$\omega = df = \sum_{i=1}^{n} D_i f \cdot dx_i$$

Since

$$f(x) = \int_0^1 \frac{d}{dt} f(tx)\, dx$$

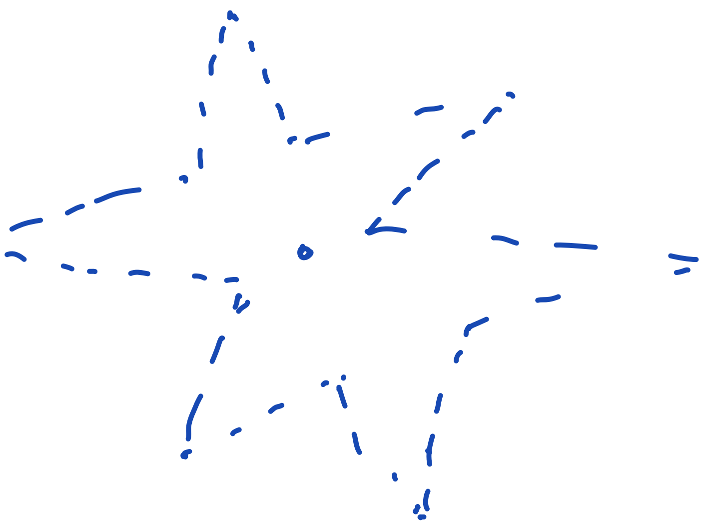
$$= \int_0^1 \sum_{i=1}^{n} D_i f(tx) \cdot x_i \, dt$$

$$= \int_0^1 \sum_{i=1}^{n} \omega_i(tx) \cdot x_i \, dt$$

This suggests:

$$I\omega(x) = \int_0^1 \sum_{i=1}^{n} \omega_i(tx) \cdot x_i \, dt$$

This is well-defined on a open set $A \subset \mathbb{R}^n$ such that if $x \in A$, then the line joining 0 to x is in A. Such an open set is called <u>star-shaped</u> with respect to 0.



It can be shown that $\omega = d(I\omega)$, provided that $d\omega = 0$.

Theorem (Poincaré Lemma). If $A \subset \mathbb{R}^n$ is an open set star-shaped with respect to 0, then every closed form on A is exact.

Proof. We will define a function I from l-forms to $(l-1)$-forms such that:

$I(0) = 0$ and $\omega = I(d\omega) + d(I\omega)$ for any form ω. Then $\omega = d(I\omega)$, if $d\omega = 0$.

Let $\omega = \sum_{i_1 < \dots < i_l} w_{i_1, \dots i_l} dx_{i_1} \wedge \dots \wedge dx_{i_l}$

Since A is star shaped, we define:

$$I\omega(x) = \sum_{i_1 < \cdots < i_\ell} \sum_{\alpha=1}^{\ell} (-1)^{\alpha-1} \left(\int_0^1 t^{\ell-1} \omega_{i_1, \ldots i_\ell}(tx)\, dt \right) x^{i_\alpha} dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_\alpha}} \wedge \cdots \wedge dx_{i_\ell}$$

Showing that

$$\omega = I(d\omega) + d(I\omega) \text{ is}$$

left as an exercise ∎

Geometric Properties

Defn. A singular n-cube in $A \subset \mathbb{R}^n$ is a continuous function $c : [0,1]^n \longrightarrow A$.

Example

(a) A singular 0-cube is an $f : \{0\} \longrightarrow A$.

(b) The standard n-cube in $\mathbb{R}^n$ is $I^n : [0,1]^n \longrightarrow \mathbb{R}^n$ defined by $I^n(x) = x, \ \forall x$.

Defn. A formal sum of the form $\sum_{i=1}^{k} a_i c_i$, where $a_i \in \mathbb{Z}$ and each c_i is a singular n-cube in A is called an **n-chain in A**.

Defn. (a) For each i, $1 \le i \le n$, we define two singular $(n-1)$-cubes $I^n_{(i,0)}$ and $I^n_{(i,1)}$ as follows:

$$I^n_{(i,0)}(x) := I^n(x_1, \dots, x_{i-1}, 0, x_i, \dots, x_{n-1})$$
$$= (x_1, \dots, x_{i-1}, 0, x_i, \dots, x_{n-1})$$

$$I^n_{(i,1)}(x) := I^n(x_1, \dots, x_{i-1}, 1, x_i, \dots, x_{n-1})$$
$$= (x_1, \dots, x_{i-1}, 1, x_i, \dots, x_{n-1})$$

$I^n_{(i,0)}$ and $I^n_{(i,1)}$ are called the $(i,0)$-face and $(i,1)$-face of I^n, respectively.

(b) We define

$$\partial I^n = \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} I^n_{(i,\alpha)}$$

(c) For a general singular n-cube $c : [0,1]^n \longrightarrow A$,

we define $c_{(i,\alpha)} = c \circ \left(I^n_{(i,\alpha)} \right)$

Then we define,

$$\partial c = \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} c_{(i,\alpha)}$$

(d) Finally, we define the <u>boundary</u> of the n-chain

$\sum a_i c_i$ by:

$$\partial\left(\sum a_i c_i\right) = \sum a_i \partial(c_i)$$

<u>Theorem</u>. If c is a chain in A, then $\partial(\partial c) = 0$. Briefly, $\partial^2 = 0$.

<u>Proof</u>. For $i \leq j$ and $x \in [0,1]^{n-2}$, we have:

$$\left(I^n_{(i,\alpha)}\right)_{(j,\beta)}(x) = I^n_{(i,\alpha)}\left(I^{n-1}_{(j,\beta)}(x)\right)$$

$$= I^n_{(i,\alpha)}(x_1, \ldots, x_{j-1}, \beta, x_j, \ldots x_{n-2})$$

$$= I^n(x_1, \ldots x_{i-1}, \alpha, x_i, \ldots, x_{j-1}, \beta, x_j, \ldots x_{n-2})$$

Similarly,

$$\left(I^n_{(j+1,\beta)}\right)_{(i,\alpha)}$$

$$= I^n(x_1, \ldots x_{i-1}, \alpha, x_i, \ldots, x_{j-1}, \beta, x_j, \ldots, x_{n-2})$$

$$\Longrightarrow \left(I^n_{(i,\alpha)}\right)_{(j,\beta)} = \left(I^n_{(j+1,\beta)}\right)_{(i,\alpha)},$$

for $i \leq j$.

Thus, it follows easily that:

$\left(C_{(i,\alpha)}\right)_{(j,\beta)} = \left(C_{(j+1,\beta)}\right)_{(i,\alpha)}$, for $i \leq j$

Now,

$$\partial(\partial c) = \partial\left(\sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} C_{(i,\alpha)}\right)$$

$$= \sum_{i=1}^{n} \sum_{\alpha=0,1} \sum_{j=1}^{n-1} \sum_{\beta=0,1} (-1)^{i+\alpha+j+\beta} \left(C_{(i,\alpha)}\right)_{(j,\beta)}$$

$$= 0 \quad \text{(check!)} \quad \blacksquare$$

<u>Remark.</u> If $\partial c = 0$, does $\exists$ a d in A such that $c = \partial d$.

<u>Answer</u>. no

Consider $c : [0,1] \longrightarrow \mathbb{R}^2 - \{0\}$ by $c(t) = \left(\cos(2\pi n t), \sin(2\pi n t)\right)$, where $n \in \mathbb{Z} - \{0\}$. Then $c(1) = c(0)$, so $\partial c = 0$. But $\exists$ no 2-chain c' in $\mathbb{R}^2 - 0$ such that $\partial c' = c$.

Stoke's Theorem

If ω is a k-form on $[0,1]^k$, then $\exists$ a unique f such that:

$$\omega = f\,dx_1 \wedge \dots \wedge dx_k$$

__Defn__. We define

$$\int_{[0,1]^k} \omega = \int_{[0,1]^k} f = \int_{[0,1]^k} f(x_1, \dots, x_k)\,dx_1 \cdots dx_k$$

If ω is a k-form on A and c is a singular k-cube in A, we define

$$\int_c \omega = \int_{[0,1]^k} c^* \omega$$

Remark (a) In particular, we have:

$$\int_{I^k} f\,dx_1 \wedge \dots \wedge dx_k = \int_{[0,1]^k} (I^k)^*(f\,dx_1 \wedge \dots \wedge dx_k)$$

$$= \int_{[0,1]^k} f(x_1, \dots, x_k)\,dx_1 \cdots dx_k$$

(b) When $k=0$, a 0-form ω is a function and $c : \{0\} \longrightarrow A$ is a singular 0-cube in A. So, we define:

$$\int_c \omega = \omega(c(0))$$

The integral ω over a k-chain $c = \sum a_i c_i$ is defined by:

$$\int_c \omega = \sum a_i \int_{c_i} \omega$$

(c) The integral of a 1-form over a 1-chain is often called a <u>line integral</u>.

If $Pdx + Qdy$ is a 1-form on $\mathbb{R}^2$ and $c:[0,1] \longrightarrow \mathbb{R}^2$ is a singular 1-cube (curve), then it can be shown that:

$$\int_c Pdx + Qdy$$

$$= \lim \sum_{i=1}^{n} \left(c_1(t_i) - c_1(t_{i-1})\right) \cdot P(c(t_i)) + \left(c_2(t_i) - c_2(t_{i-1})\right) \cdot Q(c(t_i)),$$

where $t_0, \dots, t_n$ is a partition of $[0,1]$ and the lim is taken over all partitions.

Theorem (Stoke's Theorem). If ω is a $(k-1)$-form on an open set $A \subset \mathbb{R}^n$ and c is a k-chain in A, then:

$$\int_c d\omega = \int_{\partial c} \omega .$$

Proof. Suppose that $c = I^k$ and ω is a $(k-1)$-form on $[0,1]^k$. Then ω is the sum of $(k-1)$-forms of the type:

$$f\, dx_1 \wedge \cdots \widehat{dx_i} \wedge \cdots \wedge dx_k \quad (*)$$

So it suffices to show the theorem for forms of the type $(*)$.

Note that

$$\int_{[0,1]^{k-1}} I^{k\,*}_{(j,\alpha)} \left(f\, dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_k \right)$$

$$= \begin{cases} 0 & , \text{if } j = i \\ \int_{[0,1]^k} f(x_1, \dots, \alpha, \dots, x_k)\, dx_1 \dots dx_k, & \text{if } j \neq i \end{cases}$$

Therefore,

$$\int_{\partial I^k} f\, dx_1 \wedge \dots \wedge \widehat{dx_i} \dots \wedge dx_k$$

$$= \sum_{j=1}^{k} \sum_{\alpha=0,1} (-1)^{j+\alpha} \int_{[0,1]^{k-1}} I^{k\,*}_{(j,\alpha)} \left(f\, dx_1 \wedge \dots \widehat{dx_i} \wedge \dots \wedge dx_k \right)$$

$$= (-1)^{i+1} \int_{[0,1]^k} f(x_1, \dots, 1, \dots, x_k)\, dx_1 \dots dx_k$$

$$+ (-1)^{i} \int_{[0,1]^k} f(x_1, \dots, 0, \dots x_k)\, dx_1 \dots dx_k$$

Moreover,

$$\int_{I^k} d\left(f dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_k\right)$$

$$= \int_{[0,1]^k} D_i f \, dx_i \wedge dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_k$$

$$= (-1)^{i-1} \int_{[0,1]^k} D_i f$$

By Fubini's theorem and FTC, we have

$$\int_{I^k} d\left(f dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_k\right)$$

$$= (-1)^{i-1} \int_0^1 \ldots \left(\int_0^1 D_i f(x_1, \ldots, x_k) dx_i \right) dx_1 \ldots \widehat{dx_i} \ldots dx_k$$

$$= (-1)^{i-1} \int_0^1 \ldots \int_0^1 [f(x_1, \ldots, 1, \ldots, x_k) - f(x_1, \ldots, 0, \ldots, x_k)] dx_1 \ldots \widehat{dx_i} \ldots dx_k$$

$$= (-1)^{i-1} \int_{[0,1]^k} f(x_1, \ldots 1, \ldots, x_k)\, dx_1 \ldots dx_k$$

$$+ (-1)^{i} \int_{[0,1]^k} f(x_1, \ldots, 0, \ldots x_k)\, dx_1 \ldots dx_k$$

Thus, $\displaystyle\int_{I^k} d\omega = \int_{\partial I^k} \omega$

For an arbitrary k-cube, it follows that:

$$\int_{\partial c} \omega = \int_{\partial I^k} c^* \omega .$$

Therefore,

$$\int_c d\omega = \int_{I^k} c^*(d\omega) = \int_{I^k} d(c^* \omega)$$

$$= \int_{\partial I^k} c^* \omega = \int_{\partial c} \omega .$$

Finally, if c is a k-chain $\sum a_i c_i$, then

$$\int_c d\omega = \sum a_i \int_{c_i} d\omega = \sum a_i \int_{\partial c_i} \omega$$

$$= \int_{\partial c} \omega \quad \blacksquare$$

Integration on chains

Multilinear algebra

Defn. Let V be a vector space over $\mathbb{R}$, and let $V^k = V \times \dots \times V$ be the k-fold product. A function $T : V^k \to \mathbb{R}$ is said to be multilinear if for each i with $1 \le i \le k$, we have:

(a) $T(v_1, \dots, v_i + v_i', \dots v_k)$

$$= T(v_1, \dots v_i, \dots v_k) + T(v_1, \dots, v_i' \dots v_k)$$

(b) $T(v_1, \dots, a v_i, \dots v_k)$

$$= a\,T(v_1, \dots v_i, \dots v_k)$$

<u>Defn</u>. A multilinear function $T: V^k \to \mathbb{R}$ is called a <u>k-tensor</u> on V.

<u>Remark</u>. The set of all k-tensors $\mathcal{T}^k(V)$ on V is a vector space over $\mathbb{R}$

<u>Defn</u>. For $S \in \mathcal{T}^k(V)$ and $T \in \mathcal{T}^l(V)$, we define the tensor product $S \otimes T \in \mathcal{T}^{k+l}(V)$

by :

$$S \otimes T(v_1, \dots, v_k, v_{k+1}, \dots, v_\ell)$$
$$= S(v_1, \dots, v_k) \circ T(v_{k+1}, \dots, v_\ell)$$

<u>Remark</u>. Note that

$$S \otimes T \neq T \otimes S$$

<u>Lemma</u>. Tensor product $\otimes$ satisfies the following properties.

(a) $(S_1 + S_2) \otimes T = S_1 \otimes T + S_2 \otimes T$

(b) $S \otimes (T_1 + T_2) = S \otimes T_1 + S \otimes T_2$

(c) $(aS) \otimes T = S \otimes aT = a(S \otimes T)$

(d) $(S \otimes T) \otimes U = (S \otimes T) \otimes U$

Remark

(i) The tensor products in (d) are usually denoted by $S \otimes T \otimes U$; higher products $T_1 \otimes \cdots \otimes T_r$ are defined similarly.

(ii) $\mathcal{T}^1(V) = V^*$ (dual space)

Theorem. Let $v_1, \ldots, v_n$ be a basis for V, and let $\varphi_1, \ldots, \varphi_n$ be basis for V^* so that $\varphi_i(v_j) = \delta_{ij}$. Then the set of all k-fold tensor products

$$\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} \quad 1 \le i_1, \ldots, i_k \le n$$

is a basis for $\mathcal{T}^k(V)$.

Consequently, $\dim(\mathcal{T}^k(V)) = n^k$.

<u>Proof</u>

Observe that

$$(\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})(v_{j_1}, \ldots, v_{j_k})$$

$$= \delta_{i_1, j_1} \cdots \delta_{i_k, j_k}$$

$$= \begin{cases} 1, & \text{if } j_r = i_r, \text{ for } 1 \le r \le k \\ 0, & \text{otherwise.} \end{cases}$$

If $w_1, \ldots, w_k$ are k vectors with $w_i = \sum_{j=1}^{n} a_{ij} v_j$ and $T \in \mathcal{T}^k(V)$, then:

$$T(w_1, \ldots, w_k) = \sum_{j_1, \ldots, j_k = 1}^{n} a_{1, j_1} \cdots a_{k, j_k} \, T(v_{j_1}, \ldots v_{j_k})$$

$$= \sum_{i_1,\dots i_k=1}^{n} T(v_{i_1},\dots v_{i_k}) \cdot (\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})(w_1,\dots,w_k)$$

$$\Rightarrow T = \sum_{i_1,\dots i_k=1}^{n} T(v_{i_1},\dots v_{i_k}) \cdot (\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})$$

$\Rightarrow \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}$ span $\mathcal{T}^k(V)$.

Now suppose that

$$\sum_{i_1,\dots i_k=1}^{n} a_{i_1,\dots,i_k} \cdot \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} = 0$$

Apply both sides to $(v_{j_1},\dots,v_{j_k})$, we have:

$$a_{j_1,\dots j_k} = 0 \quad \blacksquare$$

Remark. If $f: V \longrightarrow W$ is a linear transformation, then

$f^*: \mathcal{T}^k(W) \longrightarrow \mathcal{T}^k(V)$

defined by:

$$f^*T(v_1, \dots, v_k) = T(f(v_1), \dots, f(v_k))$$

for $T \in \mathcal{T}^k(W)$ and $v_1, \dots, v_k \in V$, is also a linear transformation.

Check: $f^*(S \otimes T) = f^*S \otimes f^*T$.

Examples.

(a) An inner product T on V ($T: V \times V \longrightarrow \mathbb{R}$) is a 2-tensor (i.e. $T \in \mathcal{T}^2(V)$)

that is:

(i) Symmetric: $T(v,w) = T(w,v)$ for all $v, w \in V$, and

(ii) Positive definite: $T(v,v) \geq 0$, for all $v \in V$.

Theorem. If T is an inner product on V, there exists a basis $v_1, \ldots, v_n$ for V such that $T(v_i, v_j) = \delta_{ij}$. (i.e. an orthonormal basis). Consequently, $\exists$ an isomorphism $f : \mathbb{R}^n \longrightarrow V$ such that

$T(f(x,y)) = \langle x, y \rangle$, for $x, y \in \mathbb{R}^n$.

where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^n$. In other words $f^*T = \langle , \rangle$

Proof. Let $w_1, \dots, w_n$ is a basis for V. Then define.

$$w_1' = w_1$$
$$w_2' = w_2 - \frac{T(w_1', w_2)}{T(w_1', w_1')} \cdot w_1'$$

$$w_3' = w_3 - \frac{T(w_1', w_3)}{T(w_1', w_1')} \cdot w_1' - \frac{T(w_2', w_3)}{T(w_2', w_2')} \cdot w_2'$$

$$\vdots$$

Then $T(w_i', w_j') = 0$ if $i \neq j$ and $w_i' \neq 0$ so that $T(w_i', w_i') > 0$.

Defining $v_i = \dfrac{w_i'}{|w_i|}$, the map $e_i \xmapsto{f} v_i$ is an isomorphism ∎

<u>Defn</u>. A k-tensor $\omega \in \mathcal{T}^k(V)$ is called alternating if

$$\omega(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -\omega(v_1, \dots, v_j, \dots, v_i, \dots, v_k),$$

for all $v_1, \dots, v_k \in V$.

The set of all alternating tensors is a subspace of $\mathcal{T}^k(V)$ denoted by $\Lambda^k(V)$.

We define

$$\mathrm{Alt}(T)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn} \sigma \cdot T(v_{\sigma(1)}, \dots, v_{\sigma(k)}),$$

where S_k is permutation group of $\{1, 2, \dots, k\}$

Theorem.

(1) If $T \in \mathcal{T}^k(V)$, then $\mathrm{Alt}(T) \in \Lambda^k(V)$.

(2) If $\omega \in \Lambda^k(V)$, then $\mathrm{Alt}(\omega) = \omega$.

(3) If $T \in \mathcal{T}^k(V)$, then $\mathrm{Alt}(\mathrm{Alt}(T)) = \mathrm{Alt}(T)$.

Proof

(1) Consider the transposition $(i\,j) \in S_k$, and let $\sigma' = \sigma \circ (i\,j)$ for each $\sigma \in S_k$.

Then

$$\mathrm{Alt}(T)(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, T(v_{\sigma(1)}, \ldots, v_{\sigma(j)}, \ldots, v_{\sigma(i)}, \ldots, v_{\sigma(k)})$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) T(v_{\sigma'(1)}, \dots, v_{\sigma'(i)}, \dots, v_{\sigma'(j)}, \dots, v_{\sigma'(k)})$$

$$= \frac{1}{k!} \sum_{\sigma' \in S_k} -\operatorname{sgn}(\sigma) T(v_{\sigma'(1)}, \dots, v_{\sigma'(k)})$$

$$= -\operatorname{Alt}(T)(v_1, \dots, v_k)$$

(2) If $\omega \in \Lambda^k(V)$ and $\sigma = (i,j)$, then $\omega(v_{\sigma(1)}, \dots, v_{\sigma(k)})$

$$= \operatorname{sgn}(\sigma) \cdot \omega(v_1, \dots, v_k). \quad (*)$$

Since every $\sigma \in S_k$ is a product of transpositions, $(*)$ holds for all $\sigma \in S_k$.

Therefore,

$\mathrm{Alt}(\omega)(v_1, \dots, v_k)$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \cdot \omega(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \cdot \mathrm{sgn}(\sigma) \cdot \omega(v_1, \dots, v_k)$$

$$= \omega(v_1, \dots, v_k).$$

(3) Follows from ① & ②.

<u>Note</u>. $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^\ell(V)$

$\not\Rightarrow \omega \otimes \eta \in \Lambda^{k+\ell}(V)$.

<u>Defn</u>. For $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^\ell(V)$, we define the <u>wedge product</u> by:

$$\omega \wedge \eta := \frac{(k+\ell)!}{k!\,\ell!} \mathrm{Alt}(\omega \otimes \eta)$$

Lemma. Wedge product satisfies the following properties:

(a) $(\omega_1+\omega_2)\wedge\eta = \omega_1\wedge\eta + \omega_2\wedge\eta$

(b) $\omega\wedge(\eta_1+\eta_2) = \omega\wedge\eta_1 + \omega\wedge\eta_2$

(c) $a\omega\wedge\eta = \omega\wedge a\eta = a(\omega\wedge\eta)$

(d) $\omega\wedge\eta = (-1)^{kl}\,\eta\wedge\omega$

(e) $f^*(\omega\wedge\eta) = f^*(\omega)\wedge f^*(\eta)$

Theorem

(1) If $S\in\mathcal{J}^k(V)$ and $T\in\mathcal{J}^l(V)$ and $\mathrm{Alt}(S)=0$, then

$$\mathrm{Alt}(S\otimes T) = \mathrm{Alt}(T\otimes S) = 0$$

(2) $\mathrm{Alt}(\mathrm{Alt}(\omega\otimes\eta)\otimes\theta)$

$$= \mathrm{Alt}(\omega\otimes\eta\otimes\theta)$$

$$= \mathrm{Alt}(\omega \otimes \mathrm{Alt}(\eta \otimes \theta))$$

(3) If $\omega \in \Lambda^k(V)$, $\eta \in \Lambda^l(V)$, and $\theta \in \Lambda^m(V)$, then

$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta)$$
$$= \frac{(k+l+m)!}{k!\,l!\,m!} \mathrm{Alt}(\omega \otimes \eta \otimes \theta)$$

Proof.

$$(k+l)!\ \mathrm{Alt}(S \otimes T)(v_1, \dots, v_{k+l})$$
$$= \sum_{\sigma \in S_{k+l}} \operatorname{sgn}\sigma \cdot S(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \cdot T(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)})$$

Let $G \subset S_{k+l}$ consist of all σ that fix $k+1, \dots, k+l$.

Then

$$\sum_{\sigma \in G} \operatorname{sgn} \sigma \cdot S(v_{\sigma(1)}, \dots v_{\sigma(k)}) \cdot T(v_{\sigma(k+1)} \dots v_{\sigma(k+l)})$$

$$= \Big[\sum_{\sigma' \in S_k} \operatorname{sgn} \sigma' S(v_{\sigma'(1)}, \dots, v_{\sigma'(k)})\Big] \cdot T(v_{k+1}, \dots, v_{k+l})$$

Now let $\sigma \in S_{k+l} \setminus G$, let $G \cdot \sigma_0 = \{\sigma \cdot \sigma_0 \mid \sigma \in G\}$, and $v_{\sigma_0(1)}, \dots v_{\sigma_0(k+l)} = w_1, \dots, w_{k+l}$.

Then

$$\sum_{\sigma \in G \cdot \sigma_0} \operatorname{sgn} \sigma \cdot S(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \cdot T(v_{\sigma(k+1)}, \dots v_{\sigma(k+l)})$$

$$= \Big[\operatorname{sgn} \sigma_0 \cdot \sum_{\sigma' \in G} \operatorname{sgn} \sigma' S(w_{\sigma'(1)} \dots w_{\sigma'(k)}) \cdot T(w_{k+1}, \dots, w_{k+l})$$

$$= 0 .$$

(Note that $G \cap G \cdot \sigma_0 = \emptyset$).

(2) We have

$$\mathrm{Alt}(\mathrm{Alt}(\eta \otimes \theta) - \eta \otimes \theta)$$
$$= \mathrm{Alt}(\eta \otimes \theta) - \mathrm{Alt}(\eta \otimes \theta)$$

$\Rightarrow$ By (1), we have

$$0 = \mathrm{Alt}(\omega \otimes [\mathrm{Alt}(\eta \otimes \theta) - \eta \otimes \theta])$$
$$= \mathrm{Alt}(\omega \otimes \mathrm{Alt}(\eta \otimes \theta))$$
$$- \mathrm{Alt}(\omega \otimes \eta \otimes \theta)$$

(3) $(\omega \wedge \eta) \wedge \theta$

$$= \frac{(k+l+m)!}{(k+l)!\, m!} \mathrm{Alt}((\omega \wedge \eta) \otimes \theta)$$

$$= \frac{(k+l+m)!}{(k+l)!\, m!} \frac{(k+l)!}{k!\, l!} \mathrm{Alt}(\omega \otimes \eta \otimes \theta)$$

We denote both $\omega \wedge (\eta \wedge \theta)$ and $(\omega \wedge \eta) \wedge \theta$ by $\omega \wedge \eta \wedge \theta$.

Higher-order products are denoted by $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_r$.

Theorem. The set of all $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}$ $\quad 1 \le i_1 < i_2 < \cdots < i_k \le n$ is a basis for $\Lambda^k(V)$.

Consequently,

$$\dim \Lambda^k(V) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Theorem. Let $v_1, \ldots, v_n$ be a basis for V, and let $\omega \in \Lambda^n(V)$. If $w_i = \sum_{j=1}^{n} a_{ij} v_j$ for $1 \le i \le n$, then:

$$\omega(w_1, \dots w_n) = \det(a_{ij})\,\omega(v_1, \dots, v_n).$$

Proof. Define $\eta \in \mathcal{J}^n(\mathbb{R}^n)$ by

$$\eta((a_{11}, \dots, a_{1n}), \dots, (a_{n1}, \dots, a_{nn}))$$
$$= \omega(\sum a_{1j} v_j, \dots, \sum a_{nj} v_j)$$

Then $\eta \in \Lambda^n(\mathbb{R}^n)$ and

$$\eta = \eta(e_1, \dots, e_n) \cdot \det(a_{ij})$$
$$= \omega(v_1, \dots v_n) \cdot \det(a_{ij}) \blacksquare$$

Remark. By theorem, a nonzero $w \in \Lambda^n(V)$ splits bases of V into two groups:

(a) Those with $\omega(v_1, \dots, v_n) < 0$

(b) Those with $\omega(v_1, \dots, v_n) > 0$.

Two bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are in the same group if given $w_i = \sum a_{ij} v_j$, then $\det(a_{ij}) > 0$.

<u>Defn</u>. Either of these two groups is called an <u>orientation</u> for V.

In $\mathbb{R}^n$, the <u>usual orientation</u> is $[e_1, \ldots, e_n]$.

Remark. (a) Note that $\dim \Lambda^n(\mathbb{R}^n) = 1$. In fact, det is often seen as the unique $\omega \in \Lambda^n(\mathbb{R}^n)$ such that $\omega(e_1, \ldots, e_n) = 1$

<u>Why?</u> Suppose that T is an inner product and $v_1, \ldots v_n; w_1, \ldots w_n$

are two bases which are orthonormal with respect to T with $w_i = \sum_{j=1}^{n} a_{ij} v_j$.

Then

$$\delta_{ij} = T(w_i, w_j) = \sum_{k,l=1}^{n} a_{ik} a_{jl} T(v_k, v_l)$$
$$= \sum_{k=1}^{n} a_{ik} a_{jk}.$$

$$\Rightarrow \quad A \cdot A^T = I \Rightarrow \det(A) = \pm 1.$$

By theorem, if $\omega \in \Lambda^n(V)$ satisfies $\omega(v_1, \ldots, v_n) \pm 1$, then $\omega(w_1, \ldots, w_n) = \pm 1$.

If an orientation μ for V has been given, then $\exists!\ \omega \in \Lambda^n(V)$ such that $\omega(v_1, \dots, v_n) = 1$, whenever $v_1, \dots, v_n$ is an orthonormal basis such that $[v_1, \dots, v_n] = \mu$.

<u>Defn.</u>
This unique ω is called the <u>volume element</u> of V, determined by T and μ.

<u>Example</u> det is the volume element of $\mathbb{R}^n$ with $\langle\ \rangle$ and $[e_1, \dots, e_n]$.

In fact, $|\det(v_1, \dots, v_n)|$ = volume of parallelopiped spanned by $v_1, \dots, v_n$.

Defn. Let $v_1, \ldots, v_{n-1} \in \mathbb{R}^n$ and φ is defined by

$$\varphi(w) = \det \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ w \end{pmatrix}$$

Then $\varphi \in \Lambda^1(\mathbb{R}^n)$ and $\exists!\, z \in \mathbb{R}^n$ such that

$$\langle w, z \rangle = \varphi(w) = \det \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ w \end{pmatrix}$$

This z is denoted by $v_1 \times \cdots \times v_{n-1}$ and is called the <u>cross-product of $v_1, \ldots, v_{n-1}$</u>.

Lemma (a) $v_{\sigma(1)} \times \cdots \times v_{\sigma(n-1)}$
$$= \operatorname{sgn}\sigma \cdot (v_1 \times \cdots \times v_{n-1})$$

(b) $v_1 \times \cdots \times a v_i \times \cdots v_{n-1} = a \cdot (v_1 \times \cdots \times v_n)$

(c) $v_1 \times \cdots \times (v_i + v_i') \times \cdots \times v_{n-1}$
$$= v_1 \times \cdots v_i \times \cdots v_{n-1} + v_1 \times \cdots \times v_i' \times \cdots \times v_{n-1}.$$

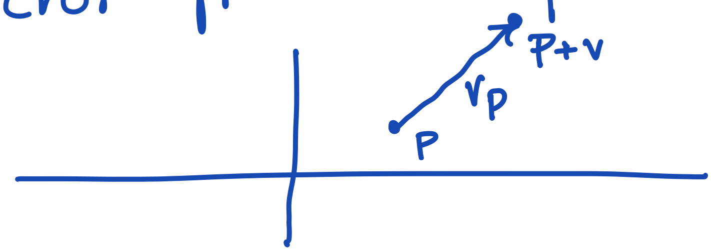
Vector fields and Differential Forms

Defn. For $p \in \mathbb{R}^n$, the <u>tangent space of $\mathbb{R}^n$ at p</u> is defined by $\mathbb{R}^n_p = \{(p, v) : v \in \mathbb{R}^n\}$.

Remark. $\mathbb{R}^n_p$ is a vector space with respect to:

$$(p,v) + (p,w) = (p, v+w)$$
$$a \cdot (p,v) = (p, av)$$

Given p and $v \in \mathbb{R}^n_p$, we write $v_p = (v, p)$ and visualize it as a vector from the point p to $p+v$



The standard inner product $\langle . \rangle$ on $\mathbb{R}^n$ induces an inner product $\langle , \rangle_p$ on $\mathbb{R}^n_p$ define by $\langle u_p, v_p \rangle_p = \langle u, v \rangle$

<u>Defn</u>. A <u>vector field</u> is a function $F : \mathbb{R}^n \longrightarrow \bigcup_{x \in \mathbb{R}^n} \mathbb{R}^n_x$ such that $F(x) \in \mathbb{R}^n_p$, for each $p \in \mathbb{R}^n$.

<u>Remark</u>.
For each $p \in \mathbb{R}^n$, $\exists$ $F_1(p), \ldots, F_n(p)$ such that

$$F(p) = \sum_{i=1}^{n} F_i(p)(e_i)_p$$

, where the F_i are the component functions.

Defn A vector field F is continuous (resp. diff) if each F_i is continuous (resp. diff).

Defn. If F, G are vector fields, and f is a function, we define:

(a) $(F+G)(p) = F(p)+G(p)$

(b) $\langle F, G\rangle(p) = \langle F(p), G(p)\rangle$

(c) $(f \cdot F)(p) = f(p)\, F(p)$

Defn. If F_i, $1 \le i \le n$, are vector fields, we define:

$$(F_1 \times \cdots \times F_{n-1})(p) = F_1(p) \times \cdots \times F_{n-1}(p)$$

Defn We define the <u>divergence</u> of a vector field F by

$$\operatorname{div}(F) = \sum_{i=1}^{n} D_i F_i$$

In symbols, if $\nabla = \sum_{i=1}^{n} D_i \cdot e_i$, then $\operatorname{div}(F) = \langle \nabla, F \rangle$

Defn. Under this symbolism, we define the <u>curl</u> of F as the vector field

$$(\nabla \times F)(p) = \begin{vmatrix} (e_1)_p & (e_2)_p & (e_3)_p \\ D_1 & D_2 & D_3 \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Defn. A function

$$\omega : \mathbb{R}^n \longrightarrow \bigcup_{x \in \mathbb{R}^n} \Lambda^k(\mathbb{R}^n_x)$$

Such that $\omega(p) \in \Lambda^k(\mathbb{R}^n_p)$, for each $p \in \mathbb{R}^n$ is called a <u>differentiable k-form</u> on $\mathbb{R}^n$

If $\varphi_1(p), \dots, \varphi_n(p)$ is a dual basis to $(e_1)_p, \dots, (e_n)_p$, then

$$\omega(p) = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k}(p) \cdot [\varphi_{i_1}(p) \wedge \dots \wedge \varphi_{i_k}(p)]$$

<u>Remark</u> (a) The operations $\omega + \eta$, $f \cdot \omega$. $\omega \wedge \eta$ are well-defined.

(b) A function f is considered to be a 0-form.

If $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable, then $Df(p) \in \Lambda^1(\mathbb{R}^n)$. So we define df by:

$$df(p)(v_p) = Df(p)(v)$$

For any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let

$$x \xmapsto{\pi_i} x_i$$

Then

$$dx_i(p)(v_p) = d\pi_i(p)v_p = D\pi_i(p)(v)$$
$$= v_i$$

(Here we view x_i as π_i)

So, $dx_1(p), \ldots, dx_n(p)$ is a dual basis to $(e_1)_p, \ldots, (e_n)_p$.

Thus, every k-form can be written as

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

Theorem. If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable, then

$$df = D_1 f \cdot dx_n + \cdots + D_n f \cdot dx_n$$

i.e. in classical notation,

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

$$\left(dx_i(p) = d\pi_i(p) \right)$$

Proof.

$$df_p(v_p) = Df(p)(v)$$

$$= \sum_{i=1}^{n} v_i D_i f(p)$$

$$= \sum_{i=1}^{n} dx_i(p) v_p \cdot D_i f(p) \blacksquare$$

Consider $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $Df(p) : \mathbb{R}^m \longrightarrow \mathbb{R}^n$. Then $f_* : \mathbb{R}^n_p \longrightarrow \mathbb{R}^m_{f(p)}$ is defined by

$$f_*(v_p) = \left(Df(p)(v)\right)_{f(p)}.$$

This linear map induces a linear map

$$f^* : \Lambda^k\left(\mathbb{R}^m_{f(p)}\right) \longrightarrow \Lambda^k\left(\mathbb{R}^n_p\right)$$

If w is a k-form on $\mathbb{R}^m$, we define a k-form f^*w on $\mathbb{R}^n$ by:

$$(f^*w)(p) = f^*\left(w(f(p))\right)$$

i.e. if $v_1, \ldots, v_k \in \mathbb{R}^n_p$, then

$$(f^*w)(p)(v_1, \ldots, v_k)$$
$$= w(f(p))\left(f_*(v_1), \ldots, f_*(v_k)\right)$$

Theorem. If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable, then:

(a) $f^*(dx_i) = \sum_{j=1}^{n} D_j f_i \cdot dx_j$

$$= \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} dx_j$$

(b) $f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2)$

(c) $f^*(g \cdot \omega) = (g \circ f) \cdot f^*\omega$

(d) $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$

Proof

(a)

$$f^*(dx_i)(p)(v_p) = dx_i(f(p))(f_* v_p)$$

$$= dx_i(f(p))\left(\sum_{j=1}^{n} v_j D_j f_1(p), \ldots, \sum_{j=1}^{n} v_j D_j f_m(p)\right)_{f(p)}$$

$$= \sum_{j=1}^{n} v_j D_j f_i(p)$$

$$= \sum_{j=1}^{n} D_j f_i(p) \cdot dx_j(p)(v_p) \blacksquare$$

Theorem. If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is differentiable, then

$$f^*(h\,dx_1 \wedge \cdots \wedge dx_n) = (h \circ f)(\det f')\,dx_1 \wedge \cdots \wedge dx_n$$

Proof. Since

$$f^*(h\,dx_1 \wedge \cdots \wedge dx_n) = (h \circ f)\, f^*(dx_1 \wedge \cdots \wedge dx_n),$$

it suffices to show that

$$f^*(dx_1 \wedge \cdots \wedge dx_n) = \det(Df)\,dx_1 \wedge \cdots \wedge dx_n$$

Let $p \in \mathbb{R}^n$ and let $A = (a_{ij}) = Df(p)$

Then

$$f^*(dx_1 \wedge \cdots \wedge dx_n)(e_1, \ldots, e_n)$$
$$= dx_1 \wedge \cdots \wedge dx_n(f_* e_1, \ldots, f_* e_n)$$

$$= dx_1 \wedge \dots \wedge dx_n \left(\sum_{i=1}^{n} a_{i1} e_i, \dots, \sum_{i=1}^{n} a_{in} e_i \right)$$

$$= \det(a_{ij}) \cdot dx_1 \wedge \dots \wedge dx_n (e_1, \dots, e_n)$$

■

<u>Defn</u> Given the k-form

$$\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k},$$

we define a $(k+1)$-form $d\omega$, the differential of ω, by

$$d\omega = \sum_{i_1 < \dots < i_k} d\omega_{i_1 \dots i_k} \wedge dx_{i_1} \wedge \dots dx_{i_k}$$

$$= \sum_{i_1 < \dots < i_k} \sum_{\alpha = 1}^{n} D_\alpha (\omega_{i_1, \dots, i_k}) \cdot dx_\alpha \wedge dx_{i_1} \wedge \dots dx_{i_k}$$

<u>Theorem</u>.

(i) $d(\omega + \eta) = d\omega + d\eta$

(ii) If ω is a k-form and η is a 1-form, then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(iii) $d(d\omega)=0$ (i.e. $d^2=0$)

(iv) If ω is a k-form on $\mathbb{R}^m$ and $f:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ is diff, then $f^*(d\omega) = d(f^*\omega)$.

<u>Defn</u>. A form ω is <u>closed</u> if $d\omega=0$ and <u>exact</u> if $\omega = d\eta$, for some η.

<u>Remark</u> (i) By theorem, every exact form is closed.

Conversely, if $\omega = P\,dx + Q\,dy$ is a 1-form in $\mathbb{R}^2$, then

$$
\begin{aligned}
d\omega &= (D_1P\,dx + D_2P\,dy)\wedge dx \\
&\quad + (D_1Q\,dx + D_2Q\,dy)\wedge dy \\
&= (D_1Q - D_2P)\,dx\wedge dy
\end{aligned}
$$

So, if $d\omega = 0$, then

$$D_1 Q = D_2 P.$$

$\exists$ a function f such that

$\omega = df = D_1 f dx + D_2 f dy$. (HW)

(ii) However, if ω is defined only on a subset of $\mathbb{R}^2$

For example, consider

$$\omega = \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$$

on $\mathbb{R}^2 - \{0\}$

Then $\omega = d\theta$, where

$$\theta(x,y) = \begin{cases} \tan^{-1}(y/x) & x, y > 0 \\ \pi + \tan^{-1}(y/x) & x < 0 \\ 2\pi + \tan^{-1}(y/x) & x > 0, y < 0 \\ \pi/2 & x = 0, y > 0 \\ 3\pi/2 & x = 0, y < 0 \end{cases}$$

which is not continuous on $\mathbb{R}^2 - \{0\}$.

If $\omega = df$, for some $f : \mathbb{R}^2 - 0 \to \mathbb{R}$,
then $D_1 f = D_1\theta$ and $D_2 f = D_2\theta$

$\Rightarrow f = \theta + c \Rightarrow f$ cannot exist.

(iii) Suppose that $\omega = \sum_{i=1}^{n} \omega_i dx_i$ is a 1-form on $\mathbb{R}^n$ and

$$\omega = df = \sum_{i=1}^{n} D_i f \cdot dx_i$$

Since

$$f(x) = \int_0^1 \frac{d}{dt} f(tx)\, dx$$

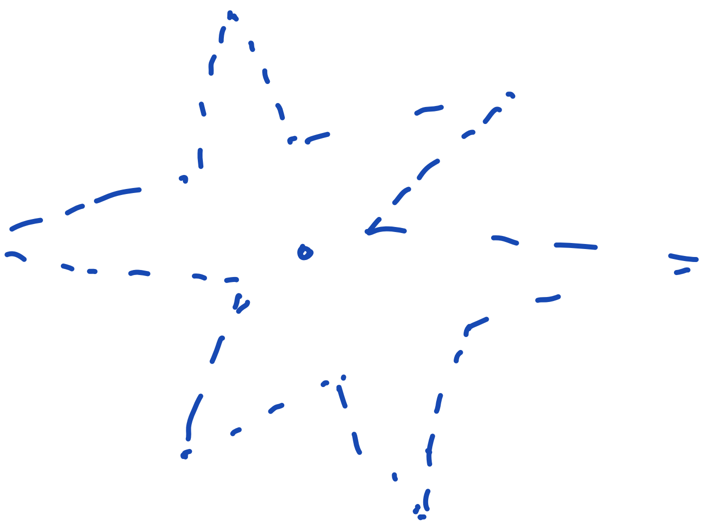
$$= \int_0^1 \sum_{i=1}^{n} D_i f(tx) \cdot x_i \, dt$$

$$= \int_0^1 \sum_{i=1}^{n} \omega_i(tx) \cdot x_i \, dt$$

This suggests:

$$I\omega(x) = \int_0^1 \sum_{i=1}^{n} \omega_i(tx) \cdot x_i \, dt$$

This is well-defined on a open set $A \subset \mathbb{R}^n$ such that if $x \in A$, then the line joining 0 to x is in A. Such an open set is called <u>star-shaped</u> with respect to 0.



It can be shown that $\omega = d(I\omega)$, provided that $d\omega = 0$.

Theorem (Poincaré Lemma). If $A \subset \mathbb{R}^n$ is an open set star-shaped with respect to 0, then every closed form on A is exact.

Proof. We will define a function I from l-forms to $(l-1)$-forms such that:

$I(0) = 0$ and $\omega = I(d\omega) + d(I\omega)$ for any form ω. Then $\omega = d(I\omega)$, if $d\omega = 0$.

Let $\omega = \sum_{i_1 < \dots < i_l} \omega_{i_1, \dots i_l} dx_{i_1} \wedge \dots \wedge dx_{i_l}$

Since A is star shaped, we define:

$$I\omega(x) = \sum_{i_1<\cdots<i_\ell} \sum_{\alpha=1}^{\ell} (-1)^{\alpha-1}$$

$$\left(\int_0^1 t^{\ell-1} \omega_{i_1,\ldots,i_\ell}(tx)\, dt \right) x^{i_\alpha}$$

$$dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_\alpha}} \wedge \cdots \wedge dx_{i_\ell}$$

Showing that

$$\omega = I(d\omega) + d(I\omega) \text{ is}$$

left as an exercise $\blacksquare$

Geometric Properties

Defn. A singular n-cube in $A \subset \mathbb{R}^n$ is a continuous function $c : [0,1]^n \longrightarrow A$.

Example

(a) A singular 0-cube is an $f : \{0\} \longrightarrow A$.

(b) The standard n-cube in $\mathbb{R}^n$ is $I^n : [0,1]^n \longrightarrow \mathbb{R}^n$ defined by $I^n(x) = x, \ \forall x$.

Defn. A formal sum of the form $\sum_{i=1}^{k} a_i c_i$, where $a_i \in \mathbb{Z}$ and each c_i is a singular n-cube in A is called an <u>n-chain in A</u>.

Defn. (a) For each i, $1 \le i \le n$, we define two singular $(n-1)$-cubes $I^n_{(i,0)}$ and $I^n_{(i,1)}$ as follows:

$$I^n_{(i,0)}(x) := I^n(x_1, \dots, x_{i-1}, 0, x_i, \dots, x_{n-1})$$
$$= (x_1, \dots, x_{i-1}, 0, x_i, \dots, x_{n-1})$$

$$I^n_{(i,1)}(x) := I^n(x_1, \dots, x_{i-1}, 1, x_i, \dots, x_{n-1})$$
$$= (x_1, \dots, x_{i-1}, 1, x_i, \dots, x_{n-1})$$

$I^n_{(i,0)}$ and $I^n_{(i,1)}$ are called the $(i,0)$-face and $(i,1)$-face of I^n, respectively.

(b) We define

$$\partial I^n = \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} I^n_{(i,\alpha)}$$

(c) For a general singular n-cube $c : [0,1]^n \longrightarrow A$, we define $c_{(i,\alpha)} = c \circ \left(I^n_{(i,\alpha)} \right)$

Then we define,

$$\partial c = \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} c_{(i,\alpha)}$$

(d) Finally, we define the <u>boundary</u> of the n-chain

$\sum a_i c_i$ by:

$$\partial\left(\sum a_i c_i\right) = \sum a_i \partial(c_i)$$

<u>Theorem</u>. If c is a chain in A, then $\partial(\partial c) = 0$. Briefly, $\partial^2 = 0$.

<u>Proof</u>. For $i \leq j$ and $x \in [0,1]^{n-2}$, we have:

$$\left(I^n_{(i,\alpha)}\right)_{(j,\beta)}(x) = I^n_{(i,\alpha)}\left(I^{n-1}_{(j,\beta)}(x)\right)$$

$$= I^n_{(i,\alpha)}(x_1, \ldots, x_{j-1}, \beta, x_j, \ldots x_{n-2})$$

$$= I^n(x_1, \ldots x_{i-1}, \alpha, x_i, \ldots, x_{j-1}, \beta, x_j, \ldots x_{n-2})$$

Similarly,

$$\left(I^n_{(j+1,\beta)}\right)_{(i,\alpha)}$$

$$= I^n(x_1, \dots x_{i-1}, \alpha, x_i, \dots, x_{j-1}, \beta, x_j, \dots, x_{n-2})$$

$$\Rightarrow \left(I^n_{(i,\alpha)}\right)_{(j,\beta)} = \left(I^n_{(j+1,\beta)}\right)_{(i,\alpha)},$$

for $i \leq j$.

Thus, it follows easily that:

$$\left(c_{(i,\alpha)}\right)_{(j,\beta)} = \left(c_{(j+1,\beta)}\right)_{(i,\alpha)}, \text{ for } i \leq j$$

Now,

$$\partial(\partial c) = \partial\left(\sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} c_{(i,\alpha)}\right)$$

$$= \sum_{i=1}^{n} \sum_{\alpha=0,1} \sum_{j=1}^{n-1} \sum_{\beta=0,1} (-1)^{i+\alpha+j+\beta} \left(C_{(i,\alpha)}\right)_{(i,\beta)}$$

$$= 0 \quad (\text{check!})$$

<u>Remark.</u> If $\partial c = 0$, does $\exists$ a d in A such that $c = \partial d$.

<u>Answer</u>. no

Consider $c : [0,1] \longrightarrow \mathbb{R}^2 - \{0\}$ by $c(t) = (\cos(2\pi n t), \sin(2\pi n t))$, where $n \in \mathbb{Z} - \{0\}$. Then $c(1) = c(0)$, so $\partial c = 0$. But $\exists$ no 2-chain c' in $\mathbb{R}^2 - 0$ such that $\partial c' = c$.

Stoke's Theorem

If ω is a k-form on $[0,1]^k$, then $\exists$ a unique f such that:

$$\omega = f\, dx_1 \wedge \dots \wedge dx_k$$

Defn. We define

$$\int_{[0,1]^k} \omega = \int_{[0,1]^k} f = \int_{[0,1]^k} f(x_1, \dots, x_k)\, dx_1 \cdots dx_k$$

If ω is a k-form on A and c is a singular k-cube in A, we define

$$\int_c \omega = \int_{[0,1]^k} c^* \omega$$

Remark (a) In particular, we have:

$$\int_{I^k} f\,dx_1 \wedge \cdots \wedge dx_k = \int_{[0,1]^k} (I^k)^*(f\,dx_1 \wedge \cdots \wedge dx_k)$$

$$= \int_{[0,1]^k} f(x_1, \ldots, x_k)\,dx_1 \cdots dx_k$$

(b) When $k=0$, a 0-form ω is a function and $c:\{0\} \longrightarrow A$ is a singular 0-cube in A. So, we define:

$$\int_c \omega = \omega(c(0))$$

The integral ω over a k-chain $c = \sum a_i c_i$ is defined by:

$$\int_c \omega = \sum a_i \int_{c_i} \omega$$

(c) The integral of a 1-form over a 1-chain is often called a <u>line integral</u>.

If $Pdx+Qdy$ is a 1-form on $\mathbb{R}^2$ and $c:[0,1] \longrightarrow \mathbb{R}^2$ is a singular 1-cube (curve), then it can be shown that:

$$\int_c Pdx+Qdy$$

$$= \lim \sum_{i=1}^{n} \left(c_1(t_i) - c_1(t_{i-1})\right) \cdot P\left(c(t_i)\right) + \left(c_2(t_i) - c_2(t_{i-1})\right) \cdot Q\left(c(t_i)\right),$$

where $t_0, \ldots, t_n$ is a partition of $[0,1]$ and the lim is taken over all partitions.

Theorem (Stoke's Theorem). If ω is a $(k-1)$-form on an open set $A \subset \mathbb{R}^n$ and c is a k-chain in A, then:

$$\int_c d\omega = \int_{\partial c} \omega .$$

Proof. Suppose that $c = I^k$ and ω is a $(k-1)$-form on $[0,1]^k$. Then ω is the sum of $(k-1)$-forms of the type:

$$f dx_1 \wedge \dots \widehat{dx_i} \wedge \dots \wedge dx_k \quad (*)$$

So it suffices to show the theorem for forms of the type $(*)$.

Note that

$$\int_{[0,1]^{k-1}} I^{k\,*}_{(j,\alpha)}\left(f\,dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_k\right)$$

$$=\begin{cases} 0 & \text{, if } j=i \\ \displaystyle\int_{[0,1]^k} f(x_1,\dots,\alpha,\dots,x_k)\,dx_1\dots dx_k, & \text{if } j\neq i \end{cases}$$

Therefore,

$$\int_{\partial I^k} f\,dx_1\wedge\cdots\wedge\widehat{dx_i}\cdots\wedge dx_k$$

$$=\sum_{j=1}^{k}\sum_{\alpha=0,1}(-1)^{j+\alpha}\int_{[0,1]^{k-1}} I^{k\,*}_{(j,\alpha)}\left(f\,dx_1\wedge\cdots\widehat{dx_i}\wedge\cdots\wedge dx_k\right)$$

$$=(-1)^{i+1}\int_{[0,1]^k} f(x_1,\dots,1,\dots,x_k)\,dx_1\dots dx_k$$

$$+(-1)^{i}\int_{[0,1]^k} f(x_1,\dots,0,\dots x_k)\,dx_1\dots dx_k$$

Moreover,

$$\int_{I^k} d(f\,dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_k)$$
$$= \int_{[0,1]^k} D_i f\, dx_i \wedge dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_k$$
$$= (-1)^{i-1} \int_{[0,1]^k} D_i f$$

By Fubini's theorem and FTC, we have

$$\int_{I^k} d(f\,dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_k)$$
$$= (-1)^{i-1} \int_0^1 \dots \left(\int_0^1 D_i f(x_1, \dots, x_k)\, dx_i \right) dx_1 \dots \widehat{dx_i} \dots dx_k$$
$$= (-1)^{i-1} \int_0^1 \dots \int_0^1 [f(x_1, \dots, 1, \dots, x_k) - f(x_1, \dots, 0, \dots, x_k)]\, dx_1 \dots \widehat{dx_i} \dots dx_k$$

$$= (-1)^{i-1} \int_{[0,1]^k} f(x_1, \ldots 1, \ldots, x_k) dx_1 \ldots dx_k$$

$$+ (-1)^i \int_{[0,1]^k} f(x_1, \ldots, 0, \ldots x_k) dx_1 \ldots dx_k$$

Thus, $$\int_{I^k} d\omega = \int_{\partial I^k} \omega$$

For an arbitrary k-cube, it follows that:

$$\int_{\partial c} \omega = \int_{\partial I^k} c^* \omega .$$

Therefore,

$$\int_c d\omega = \int_{I^k} c^*(d\omega) = \int_{I^k} d(c^* \omega)$$

$$= \int_{\partial I^k} c^* \omega = \int_{\partial c} \omega .$$

Finally, if c is a k-chain $\sum a_i c_i$, then

$$\int_c d\omega = \sum a_i \int_{c_i} d\omega = \sum a_i \int_{\partial c_i} \omega$$

$$= \int_{\partial c} \omega \blacksquare$$